On Hill's Worst-Case Guarantee for Indivisible Bads

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Abstract

When allocating objects among agents with equal rights, people often evaluate the fairness of an allocation rule by comparing their received utilities to a benchmark *share* – a function only of her own valuation and the number of agents. This share is called a *guarantee* if for any profile of valuations there is an allocation ensuring the share of every agent. When the objects are indivisible goods, Budish [J. Political Econ., 2011] proposed MaxMinShare, i.e., the least utility of a bundle in the best partition of the objects, which is unfortunately not a guarantee. Instead, an earlier pioneering work by Hill [Ann. Probab., 1987] proposed for a share the worst-case MaxMinShare over all valuations with the same largest possible single-object value. Although Hill's share is more conservative than the MaxMinShare, it is an actual guarantee and its computation is elementary, unlike that of the MaxMinShare which involves solving an NP-hard problem. We apply Hill's approach to the allocation of indivisible bads (objects with disutilities or costs), and characterise the tight closed form of the worst-case MinMaxShare for a given value of the worst bad. We argue that Hill's share for allocating bads is effective in the sense of being close to the original MinMaxShare value, and there is much to learn about the guarantee an agent can be offered from the disutility of her worst single bad. Furthermore, we prove that the monotonic cover of Hill's share is the best guarantee that can be achieved in Hill's model for all allocation instances.

1 Introduction

The task is to fairly allocate a given pile of indivisible objects among agents with equal rights but different preferences. Since the very beginning of the fair division literature (Steinhaus, 1949), allocation rules have been evaluated in part by the worst-case utility they guarantee to each participant over all possible utility profiles of the other agents. The higher the guarantee the safer it is for an agent clueless about the others' utilities and actions to participate in the allocation process defined by the rule.

Formally, the *guarantee* offered by an allocation rule is a mapping from any utility function to the corresponding worst-case utility for an agent. This function only depends upon the number of other agents (but not on the particulars of the agents) and the domain where their utilities come from. When the objects are divisible and desirable (i.e., *goods*), and utilities are additive, the optimal (largest feasible) guarantee is $\frac{1}{n} \cdot v_i(M)$, where M is the set of goods, n is the number of agents, and i is a generic agent with utility function v_i Dubins and Spanier (1961). But in all the important practical contexts where the objects are indivisible while utilities remain additive, the search for maximal guarantees (those that cannot be improved over the entire domain of utilities) cannot be that simple. The difficulty is obvious when we consider the "one diamond and several worthless rocks" example: unless we throw away the diamond, all agents but one end up with a negligible fraction of $v_i(M)$.

To capture exactly the diamond effect when indivisible goods are distributed, the concept of MaxMinShare (Budish, 2011) has been intensely studied over the last decade (Amanatidis et al., 2017; Kurokawa et al., 2018; Huang and Lu, 2021). An agent's MaxMinShare is motivated by an imaginary divide-and-choose experiment: the agent gets the chance to partition the objects into n bundles, but is the last one to choose one bundle. Then, the agent's MaxMinShare is the utility of her worst share in the best n-partition of the objects. MaxMinShare bears some disadvantages. On the one hand, the definition is not trivial and computing its value involves solving an NP-hard problem. On the other hand, in some rare cases, the MaxMinShare is not a feasible guarantee (Procaccia and Wang, 2014); so far the best-known approximation is that a $(\frac{3}{4} + o(1))$ fraction of MaxMinShare can be guaranteed and implemented in polynomial time (Ghodsi et al., 2018; Garg and Taki, 2021).

Back to 1980s, Hill (1987) also investigated how the indivisibility of the objects affect the agent's guaranteed share by restricting attention to additive utility functions v such that $v(M) = 1$ (without loss of generality) and the most valuable object of v is worth α, $0 < \alpha < 1$; we write $\mathcal{V}(\alpha)$ for this subdomain of additive valuations. Hill proposed to study the worst-case MaxMinShare among all valuations in $\mathcal{V}(\alpha)$, which is referred to as the *Hill's share* throughout this paper. In (Hill, 1987), Hill computed for every $n \geq 2$ a function $V_n : [0, 1] \to [0, \frac{1}{n}]$, which lower-bounds Hill's share. By definition, $V_n(\alpha)$ is also a lower bound on the MaxMinShare of every utility in $\mathcal{V}(\alpha)$. Depending on α the guarantee $V_n(\alpha)$ may or may not improve upon the $\frac{3}{4}$-approximate MaxMinShare guarantee, but its great advantage is that whether a given allocation meets the guarantee for a given utility is immediately verifiable. Furthermore, Hill proved that if every agent's utility is in $\mathcal{V}(\alpha)$, it is always possible to simultaneously give each agent a share worth at least $V_n(\alpha)$, i.e., $V_n(\cdot)$ is a guarantee. Markakis and Psomas (2011) proved a stronger result: the share $V_n(\alpha_i)$ where $\max_{e \in M} v_i(e) = \alpha_i$ is a bona fide guarantee over the full domain of additive and nonnegative utilities. Moreover, an allocation implementing these individual guarantees can be computed in polynomial time. Gourvès et al. (2015) found that $V_n(\alpha)$ is not the tight characterisation of Hill's share and proved a tighter function. An interesting fact is that the tight function is not monotone in α, but its exact computation is still open.

All the aforementioned work, as well as the majority of fair division literature, focuses on the allocation of goods, and the mirror problem of bads (undesirable objects like chores, liabilities when a partnership is dissolved, etc.; see Lenstra et al. (1990)) is not as well understood as that of goods, which motivates the current work.

1.1 Our Problem and Results

We apply Hill's approach to the allocation of indivisible bads and prove a set of results parallel to those just mentioned. The diamond effect now becomes the "chore from hell" effect where the *dis*utility is concentrated in a single bad, and now $\mathcal{V}(\alpha)$ collects all disutility functions where the value of the worst bad equals α, maintaining the normalisation $v(M) = 1$.

Our results for bads resemble those just mentioned for goods, and in addition, they make the connection between Hill's share and MinMaxShare (*the largest disutility of a share in the best partition of the bads*). To be more precise, we compute first the tight characterisation of Hill's share, refined to problems with a given number m of bads, i.e., the exact upper bound $\Delta_n^{\oplus}(\alpha; m)$ of the MinMaxShare in the domain $\mathcal{V}(\alpha; m)$, where $\mathcal{V}(\alpha; m)$ contains the valuations over m objects with the highest disutility being α. This result is stated in Theorem 1. If m is not restricted, i.e., $\mathcal{V}(\alpha) = \bigcup_m \mathcal{V}(\alpha; m)$ and $\Delta_n^{\oplus}(\alpha) = \max_m \Delta_n^{\oplus}(\alpha; m)$, we illustrate the function $\Delta_n^{\oplus}(\alpha)$ for $n = 2, 3$ in Fig. 1. Just like Gourvès et al. (2015) observed for the problem of goods, this function is not monotone in α. In passing, we tighten the bounds proposed by Hill (1987) and Gourvès et al. (2015) for the worst-case MaxMinShare in the two-agent problem of goods; see Remark 1.

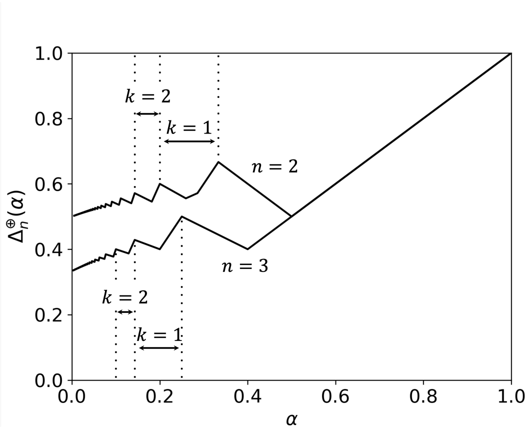


Figure 1: Hill's share $\Delta_n^{\oplus}(\alpha)$ when $n = 2$ and 3 and m is not restricted.

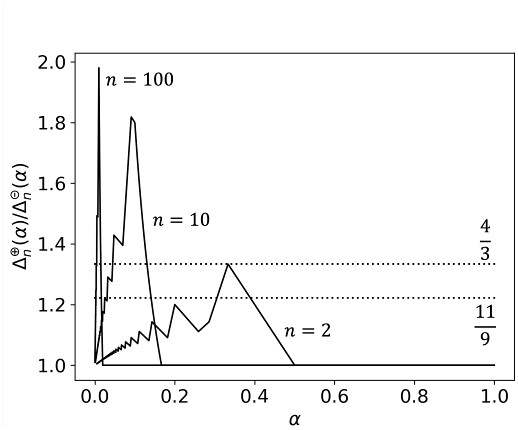


Figure 2: The ratio between the upper and lower bounds of the MinMaxShare of valuations in $\mathcal{V}(\alpha)$. 4/3 and 11/9 are two fractions of the MinMaxShare known to be achievable.

Compared to the MinMaxShare, Hill's share $\Delta_n^{\oplus}(\alpha; m)$ is immediately verifiable, whereas deciding whether (a multiple of) the MinMaxShare is met at a given allocation involves solving an NP-hard problem. Moreover, the function $\alpha \to \Delta_n^{\oplus}(\alpha; m)$ relating the guaranteed share to the disutility of the worst bad (relative to total disutility) is a transparent hard design constraint of which all participants should be aware. Although $\Delta_n^{\oplus}(\alpha; m)$ seems more conservative than the MinMaxShare of a specific disutility function, we argue that $\Delta_n^{\oplus}(\alpha; m)$ is approximately as effective as MinMaxShare. First, $\Delta_n^{\oplus}(\alpha; m)$ is at most twice the MinMaxShare of every disutility in $\mathcal{V}(\alpha; m)$. We plot the exact ratio of $\Delta_n^{\oplus}(\alpha)$ and the best MinMaxShare of disutilities in $\mathcal{V}(\alpha)$ for every α in Fig. 2 when $n = 2, 10$ and 100. As we can see, although the largest ratio may reach 2 (only happens when n is large), for most values of α, the ratio is not far from 1. In particular, $\Delta_n^{\oplus}(\alpha)$ outperforms the fractions of the MinMaxShare known to be implementable ($\frac{4}{3}$ by Barman and Krishnamurthy (2020) and $\frac{11}{9}$ by Huang and Lu (2021)) for most α no matter what values n has. Besides the above worst-case comparison, in Section 5, we conduct numerical experiments with synthetic and real-world data to illustrate the real distances between Hill's share and MinMaxShare. The experiments show that Hill's share is actually very close to (e.g., within 1.1 fraction of) the MinMaxShare for the majority of the instances.

Finally, we obtain the main result of this work – a counterpart for bads of Hill's guarantee for goods improved by Markakis and Psomas (2011). Letting $V_n(\alpha; m)$ denote the monotonic cover of $\Delta_n^{\oplus}(\alpha; m)$ with respect to α, Theorem 2 shows that the share $V_n(\alpha_i; m)$ is a guarantee over the full domain of additive disutilities with m bads. We also provide an algorithm to implement this guarantee in polynomial time. To the best of our knowledge no other similarly simple guarantee for allocating bads has been identified.

1.2 More Relevant Literature

The properties known as proportionality up to one object (Prop1) and up to any object (PropX) offer different relaxations of the equal share $\frac{1}{n}v_i(M)$ when the objects (goods or bads) are indivisible Moulin (2019); Aziz et al. (2020). These relaxations require the equal share to be satisfiable if at most one object is added or removed. Like the Hill's guarantees for goods or bads we discuss here, they are immediately verifiable, but unlike these they are not always preserved by Pareto improvements, a serious limitation of their implementation. They also do not provide agents with any guaranteed utility or disutility. The same remark applies to the popular ex-post tests no envy up to one object, or up to any object Lipton et al. (2004); Budish (2011); Caragiannis et al. (2019). Another easily verifiable test is the truncated proportional share (ATP) bound of Babaioff et al. (2022), but unlike Hill's guarantees, it improves upon the MaxMinShare for goods so it is not a feasible guarantee.

Intuitively, the allocation of undesirable bads is the mirror image of that of goods. However, adapting the results is not a simple matter of switching signs. For instance when objects are indivisible the approximations Prop1 and PropX behave quite differently for goods and bads Moulin (2019); Aziz et al. (2020).[1] Our results confirm this general observation: in our case the general allure of the critical functions $\Delta_n^{\oplus}$ is the same for goods and for bads, but the details and the proofs are quite different. We refer the readers to recent surveys by Moulin (2019) and Aziz et al. (2022) for a more detailed discussion on the fair division of indivisible goods and bads. In particular, Aziz et al. (2022) explicitly listed computing Hill's guarantee for bads as an open problem.

2 Preliminaries

For any positive integer k, let $[k] = \{1, \ldots, k\}$. We consider allocating m indivisible objects, denoted by $M = [m]$, among n agents, and let $\mathcal{A}dd(M)$ be the domain made of the nonnegative additive disutility functions v on object set M, normalised without loss of generality, as follows

$$v(S) = \sum_{e \in S} v(\{e\}) \text{ for all } S \subseteq M \text{ and } v(M) = 1.$$

Following the convention of the literature, disutility functions are also called valuations. For simplicity, we write $v(e)$ to represent $v(\{e\})$ for each $e \in M$. For any $\alpha \in [0, 1]$, the subdomain $\mathcal{V}(\alpha; m) \subseteq \mathcal{A}dd(M)$ is defined by the property $\max_{e \in M} v(e) = \alpha$ and $\mathcal{U}(\alpha; m)$ by $v(e) \leq \alpha$ for all $e \in M$. According to the definitions, $\mathcal{V}(\alpha; m) \subseteq \mathcal{U}(\alpha; m)$ for any valid pair of α and m. Note that, since the functions are all normalised, $\mathcal{V}(\alpha; m)$ is only well defined if $\alpha \times m \geq 1$, equivalently for $m \geq m_* = \lceil \frac{1}{\alpha} \rceil$ (the upper integer part of $\frac{1}{\alpha}$).

An allocation, denoted by $\mathbf{A} = (A_1, \ldots, A_n)$, is a partition of M into n disjoint subsets of objects; note that some of these subsets can be empty. The set of all allocations is denoted by $\mathcal{X}_n(M)$. The MinMaxShare (MMS), when there are n agents, of the disutility $v \in \mathcal{A}dd(M)$ is defined as

$$\mathsf{MMS}_n(v) = \min_{\mathbf{A} \in \mathcal{X}_n(M)} \max_{1 \leq \ell \leq n} v(A_\ell).$$

[1] See also Bogomolnaia et al. (2019) for the competitive equilibrium from equal incomes when objects are divisible.

We next define the upper and lower bounds of MinMaxShare among all disutilities in $\mathcal{V}(\alpha; m)$,

$$\Delta_n^{\oplus}(\alpha; m) = \max_{v \in \mathcal{V}(\alpha; m)} \mathsf{MMS}_n(v); \text{ and}$$
$$\Delta_n^{\ominus}(\alpha; m) = \min_{v \in \mathcal{V}(\alpha; m)} \mathsf{MMS}_n(v).$$

The upper bound $\Delta_n^{\oplus}(\alpha; m)$ (i.e., the worst-case MinMaxShare) is called *Hill's share*, and we use these terms interchangeably in this paper.

It is not difficult to obtain the below formula of $\Delta_n^{\ominus}(\alpha; m)$, whose formal proof is in the appendix.

Lemma 1 *Given $0 < \alpha < 1$, $n \geq 2$, and $m \geq \lceil \frac{1}{\alpha} \rceil$, $\Delta_n^{\ominus}(\alpha; m)$ is as follows:*

$$\Delta_n^{\ominus}(\alpha; m) = \begin{cases} \alpha, & \text{if } \alpha > \frac{1}{n}, \\ \frac{1}{n}, & \text{if } \alpha = \frac{1}{kn}, \text{ or } \frac{1}{(k+1)n} < \alpha < \frac{1}{kn} \text{ and } m \geq kn + n \\ k\alpha + \frac{1 - kn\alpha}{m - kn}, & \text{if } \frac{1}{(k+1)n} < \alpha < \frac{1}{kn} \text{ and } m \leq kn + n - 1 \end{cases}$$

for some integer $k \geq 1$.

Computing $\Delta_n^{\oplus}(\alpha; m)$ is non-trivial, as shown in Section 3, but the following lemma, proved in the appendix, presents two simple properties.

Lemma 2 *(1) $\Delta_n^{\oplus}(\alpha; m)$ is weakly decreasing in n; (2) $\Delta_n^{\oplus}(\alpha; m)$ is weakly increasing in m from $\lceil \frac{1}{\alpha} \rceil$ to $\lceil \frac{2}{\alpha} \rceil - 1$ and constant thereafter.*

By the second property in Lemma 2, and also following Hill (1987); Markakis and Psomas (2011); Gourvès et al. (2015), we also consider the case when m is not restricted, or equivalently, $m = \infty$. Let $\mathcal{V}(\alpha) = \bigcup_m \mathcal{V}(\alpha, m)$ and $\mathcal{U}(\alpha) = \bigcup_m \mathcal{U}(\alpha; m)$. Accordingly, we have

$$\Delta_n^{\oplus}(\alpha) = \max_{v \in \mathcal{V}(\alpha)} \mathsf{MMS}_n(v); \text{ and}$$
$$\Delta_n^{\ominus}(\alpha) = \min_{v \in \mathcal{V}(\alpha)} \mathsf{MMS}_n(v).$$

By Lemma 1, $\Delta_n^{\ominus}(\alpha) = \max\{\alpha, 1/n\}$.

Hill's share $\Delta_n^{\oplus}(\alpha; m)$ (and $\Delta_n^{\oplus}(\alpha)$) behave much like the MinMaxShare in the following senses. First, for any $v \in \mathcal{V}(\alpha; m)$ there is an allocation $(A_1, \ldots, A_n)$ such that $v(A_i) \leq \Delta_n^{\oplus}(\alpha; m)$ for all i. This follows from the definition of the MinMaxShare plus that $\Delta_n^{\oplus}(\alpha; m)$ is an upper bound of the MinMaxShare. Second, the *max* in the definition of $\Delta_n^{\oplus}(\alpha; m)$ is achieved by some $v^* \in \mathcal{V}(\alpha; m)$; that is, $\Delta_n^{\oplus}(\alpha; m) = \mathsf{MMS}_n(v^*)$. This is because $\mathcal{V}(\alpha; m)$ is a compact set and all the functions are continuous. Then we know that for any allocation $(B_1, \ldots, B_n)$ there is some i such that $v^*(B_i) \geq \Delta_n^{\oplus}(\alpha; m)$. Note that these two facts have nothing to do with what the function $\Delta_n^{\oplus}(\alpha; m)$ actually looks like and they can be easily adapted to $\Delta_n^{\oplus}(\alpha)$.

3 Characterising Hill's Share

3.1 Main Result

We now characterise Hill's share, i.e., the exact upper bound of the MinMaxShare values, $\Delta_n^{\oplus}(\alpha; m)$ and $\Delta_n^{\oplus}(\alpha)$. For any integers $n \geq 2$ and $k \geq 0$, define the following real intervals:

$$D(n, k) = \left(\frac{1}{kn + n + 1}, \frac{k+2}{n(k+1)^2 + k + 2} \right]$$
$$I(n, k) = \left(\frac{k+2}{n(k+1)^2 + k + 2}, \frac{1}{kn + 1} \right].$$

It is not hard to check that all the intervals are well-defined, non-overlapping, and $\bigcup_{k\geq 0}(D(n,k)\cup I(n,k)) = (0,1]$.

Our first main theorem gives the tight characterisation of Hill's share.

Theorem 1 *For any $0 < \alpha < 1$, $n \geq 2$, and $m \geq \lceil \frac{1}{\alpha} \rceil$,*

$$\Delta_n^{\oplus}(\alpha; m) = \begin{cases} \frac{k+2}{k+1} \cdot \frac{1-\alpha}{n}, & \text{if } \alpha \in D(n,k) \text{ and } m \geq kn + n + 1, \\ (k+1)\alpha, & \text{if } \alpha \in D(n,k) \text{ and } m \leq kn + n, \\ (k+1)\alpha, & \text{if } \alpha \in I(n,k) \end{cases} \tag{1}$$

for any integers $n \geq 2$ and $k \geq 0$ except $n = 2$ and simultaneously $k = 1$. If $n = 2$ and $k = 1$, $\Delta_2^{\oplus}(\frac{1}{3}; 3) = \frac{2}{3}$, $\Delta_2^{\oplus}(\alpha; 4) = 2\alpha$ for $\alpha \in [\frac{1}{4}, \frac{1}{3}]$, and $\Delta_2^{\oplus}(\alpha; 5)$ is as follows:

$$\Delta_2^{\oplus}(\alpha; 5) = \begin{cases} \frac{3-3\alpha}{4}, & \text{if } \alpha \in (\frac{1}{5}, \frac{3}{11}], \\ 2\alpha, & \text{if } \alpha \in (\frac{3}{11}, \frac{1}{3}], \end{cases} \tag{2}$$

and for $m \geq 6$,

$$\Delta_2^{\oplus}(\alpha; m) = \begin{cases} \frac{3-3\alpha}{4}, & \text{if } \alpha \in (\frac{1}{5}, \frac{7}{27}] \\ \alpha + \frac{2-2\alpha}{5}, & \text{if } \alpha \in (\frac{7}{27}, \frac{2}{7}] \\ 2\alpha, & \text{if } \alpha \in (\frac{2}{7}, \frac{1}{3}]. \end{cases} \tag{3}$$

Theorem 1 directly implies the result when the number of objects is not restricted, as shown in the following corollary.

Corollary 1 *For any $0 < \alpha < 1$, $n \geq 2$, $\Delta_n^{\oplus}(\alpha) = \max_{m \geq \lceil \frac{1}{\alpha} \rceil} \Delta_n^{\oplus}(\alpha; m)$.*

Actually, Corollary 1 is a special case of Theorem 1 when m is sufficiently large (e.g., $m \geq \lceil \frac{2}{\alpha} \rceil - 1$ by Lemma 2). Recall we illustrated $\Delta_2^{\oplus}(\alpha)$ and $\Delta_3^{\oplus}(\alpha)$ in Fig. 1. We observe two interesting and somewhat unintuitive facts about Theorem 1. First, $\Delta_n^{\oplus}(\cdot)$ is not monotone in α, just like Gourvès et al. (2015) observed for the problem with goods. To characterise $\Delta_n^{\oplus}(\alpha; m)$, we want to understand the worst-case disutility in $\mathcal{V}(\alpha; m)$, for which the objects can be hardly partitioned into bundles with similar disutilities. Intuitively, when the single-object disutility gets larger, it becomes harder to find such a balanced partition. However, this turns out to be imprecise. Second, the case of $n = 2$ makes a difference from $n \geq 3$. When $n = 2$ and $k = 1$, there are three steps in $\Delta_n^{\oplus}(\cdot)$: the worst-case MinMaxShare has two increasing intervals with different slops following a decreasing interval. For all the other values of n and k, there are two intervals with one decreasing and the other increasing.

Remark 1 *When $n = 2$ the problem of bads and that of goods are the same, since maximising the minimum bundle by partitioning the objects into two bundles is equivalent to minimising the maximum bundle. For $n = 2$, Gourvès et al. (2015) provided a lower bound of the MaxMinShare for goods which is not tight. It can be verified that $1 - \Delta_2^{\oplus}(\alpha)$ is strictly larger than their bound when $\alpha \in (\frac{1}{5}, \frac{3}{10})$ (Definition 2 in (Gourvès et al., 2015)). Thus, as a byproduct, Corollary 1 improves the result in Gourvès et al. (2015) for goods with $n = 2$ by giving the tight worst-case bound, i.e.,*

$$\min_{v \in \mathcal{V}(\alpha)} \max_{\mathbf{A} \in \mathcal{X}_2(M)} \min_{1 \leq \ell \leq 2} v(A_\ell) = 1 - \Delta_2^{\oplus}(\alpha).$$

In Remark 2, we show how to extend this result to two non-identical disutilities.

3.2 Roadmap for the Proof of Theorem 1

As we have discussed, after m reaches a certain value (e.g., $m \geq \lceil \frac{2}{\alpha} \rceil - 1$ by Lemma 2), Hill's share does not increase anymore, and thus Corollary 1 is a special case of Theorem 1 when m is sufficiently large. Therefore, in this subsection, we first prove Corollary 1, and in the appendix, we carefully discuss Hill's share when m is not sufficiently large, which will complete the proof of Theorem 1 accordingly. Further, we also defer the proof of case $n = 2$ and $k = 1$ to the appendix, which makes a difference from the other cases and requires a more involved analysis.

We prove Corollary 1 by contradiction, and assume that there exists a disutility $v \in \mathcal{V}(\alpha)$ whose MinMaxShare is larger than $\Delta_n^{\oplus}(\alpha)$. Let $\mathbf{A} = (A_1, \ldots, A_n)$ be a lexicographical MinMax allocation of v; that is, the largest disutility of bundles in $\mathbf{A}$ is the minimised over all allocations, and among these allocations the second largest disutility is minimised, and so on. Without loss of generality, assume $v(A_1) \geq \cdots \geq v(A_n)$ and $v(A_1) = \mathsf{MMS}_n(v) > \Delta_n^{\oplus}(\alpha)$. Let E_α denote the subset of objects whose disutilities are exactly α, i.e., $E_\alpha = \{e \in M \mid v(e) = \alpha\}$. It can be verified that $\Delta_n^{\oplus}(\alpha) \geq (k+1)\alpha$ (this is also illustrated in Fig. 1), which gives $v(A_1) > (k+1)\alpha$. Moreover, since $v(e) \leq \alpha$ for any $e \in M$, $|A_1| \geq k+2$. We have the following property.

Claim 1 *Letting j be an agent in $N \backslash \{1\}$, for any $S_1 \subseteq A_1$ and $S_j \subseteq A_j$ such that $v(S_1) > v(S_j)$, $v(S_1) - v(S_j) \geq v(A_1) - v(A_j)$.*

Proof. For the sake of contradiction, we assume that there exist $S_1' \subseteq A_1$ and $S_{j'}' \subseteq A_{j'}$ such that $v(S_1') > v(S_{j'}')$ and $v(S_1') - v(S_{j'}') < v(A_1) - v(A_{j'})$. Then we construct another allocation $\mathbf{B} = (B_1, \ldots, B_n)$ by exchanging S_1' and $S_{j'}'$, i.e., $B_1 = A_1 \setminus S_1' \cup S_{j'}'$, $B_{j'} = A_{j'} \setminus S_{j'}' \cup S_1'$ and $B_j = A_j$ for any $j \in N \backslash \{1, j'\}$. It follows that $v(B_1) < v(A_1)$, $v(B_{j'}) < v(A_1)$ and $v(B_j) = v(A_j)$ for any $j \in N \setminus \{1, j'\}$, which contradicts the assumption that $\mathbf{A}$ is a lexicographical MinMax of v. ■

The contraposition of Claim 1 gives the following.

Claim 2 *Letting j be an agent in $N \setminus \{1\}$, for any $S_1 \subseteq A_1$ and $S_j \subseteq A_j$ such that $v(A_j \setminus S_j \cup S_1) < v(A_1)$, $v(S_j) \geq v(S_1)$.*

As a warm-up, we start from the case with large α, where $k = 0$, and distinguish two subcases depending on the domain of α.

Case 1: $n \geq 2$ **and** $k = 0$

Subcase 1.1: $\alpha \in D(n, 0)$

When $\alpha \in D(n, 0)$, $\frac{1}{n+1} < \alpha \leq \frac{2}{n+2}$ and $v(A_1) > \Delta_n^{\oplus}(\alpha) = \frac{2-2\alpha}{n}$. If $E_\alpha \cap A_1 \neq \emptyset$, there exists $e^* \in A_1$ such that $v(e^*) = \alpha < v(A_1)$. Then Claim 2 gives a lower bound of $v(A_j)$ for any $j \in N \setminus \{1\}$, i.e., $v(A_j) \geq v(e^*) = \alpha$. Summing up these lower bounds leads to the following contradiction

$$1 = \sum_{j \in N} v(A_j) > \frac{2 - 2\alpha}{n} + (n-1) \cdot \alpha = \frac{(n+1)(n-2)\alpha + 2}{n} > 1,$$

where the last inequality is because $\alpha > \frac{1}{n+1}$.

Therefore, $E_\alpha \cap A_1 = \emptyset$. Then by the definition of $\mathcal{V}(\alpha)$, there must exist $j' \in N \setminus \{1\}$ such that $E_\alpha \cap A_{j'} \neq \emptyset$, and thus $v(A_{j'}) \geq \alpha$. Recall that $|A_1| \geq k + 2 = 2$, this implies there exists $S \subseteq A_1$ such that $v(A_1) > v(S) \geq \frac{1}{2} v(A_1) > \frac{1-\alpha}{n}$. According to Claim 2, $v(A_j) \geq v(S) > \frac{1-\alpha}{n}$ holds for any $j \in N \setminus \{1, j'\}$. As a result,

$$1 = \sum_{j \in N} v(A_j) > \frac{2 - 2\alpha}{n} + \alpha + (n-2) \cdot \frac{1 - \alpha}{n} = 1,$$

which is also a contradiction. Therefore, $v(A_1) > \Delta_n^{\oplus}(\alpha)$ never holds when $\alpha \in D(n, 0)$.

For the other direction, the disutility function for this subcase (see Table 1) contains one object with disutility α and n objects with disutility $\frac{1-\alpha}{n}$. Since $\frac{1}{n+1} < \alpha \leq \frac{2}{n+2}$, it follows that $\frac{1-\alpha}{n} < \alpha \leq 2 \cdot \frac{1-\alpha}{n}$. Clearly, the MinMaxShare of this disutility function is $2 \cdot \frac{1-\alpha}{n} = \Delta_n^{\oplus}(\alpha)$.

Object Disutility	Quantity
α	1
$\frac{1-\alpha}{n}$	n

Table 1: Disutility function for Subcases 1.1 and 1.2.

Subcase 1.2: $\alpha \in I(n, 0)$

When $\alpha \in I(n, 0)$, by similar reasonings, we can show that $v(A_1) > \Delta_n^{\oplus}(\alpha)$ does not hold, either. In this subcase, $\frac{2}{n+2} < \alpha \leq 1$ and $\Delta_n^{\oplus}(\alpha) = \alpha$. If $E_\alpha \cap A_1 \neq \emptyset$, there exists $e^* \in A_1$ such that $v(e^*) = \alpha < v(A_1)$ and Claim 2 gives a lower bound of $v(A_j)$ for any $j \in N \setminus \{1\}$, i.e., $v(A_j) \geq v(e^*) = \alpha$. Summing up these lower bounds leads to the following contradiction

$$1 = \sum_{j \in N} v(A_j) > n\alpha > \frac{2n}{n+2} \geq 1,$$

where the last inequality is because $n \geq 2$.

Therefore, it must hold that $E_\alpha \cap A_1 = \emptyset$ and moreover, there exists $j' \in N \setminus \{1\}$ with $E_\alpha \cap A_{j'} \neq \emptyset$. Thus, $v(A_{j'}) \geq \alpha$. Since $|A_1| \geq k + 2 = 2$, there exists $S \subseteq A_1$ such that $v(A_1) > v(S) \geq \frac{1}{2} v(A_1) > \frac{\alpha}{2}$. According to Claim 2, $v(A_j) \geq v(S) > \frac{\alpha}{2}$ holds for any $j \in N \setminus \{1, j'\}$. As a result, we have

$$1 = \sum_{j \in N} v(A_j) > \alpha + \alpha + (n-2) \cdot \frac{\alpha}{2} = \frac{n+2}{2}\alpha > 1,$$

which is also a contradiction.

For the other direction, the disutility function for this subcase also contains one object with disutility α and n objects with disutility $\frac{1-\alpha}{n}$ (see Table 1). Since $\frac{2}{n+2} < \alpha \leq 1$, it follows that $2 \cdot \frac{1-\alpha}{n} < \alpha \leq 1$. Clearly, the MinMaxShare of this disutility function is $\alpha = \Delta_n^{\oplus}(\alpha)$. Up to here, the proof regarding the case of $k = 0$ is completed.

Next, we consider the general case of $k \geq 1$ excluding $n = 2$ and $k = 1$.

Case 2: $n \geq 3$ **and** $k \geq 1$ **or** $n \geq 2$ **and** $k \geq 2$

For this case, we again start with the subcases when $\alpha \in D(n, k)$. Recall that when $\alpha \in D(n, k)$, $\alpha \in (\frac{1}{(k+1)n+1}, \frac{k+2}{n(k+1)^2+k+2}]$ and $v(A_1) > \Delta_n^{\oplus}(\alpha) = \frac{k+2}{k+1} \cdot \frac{1-\alpha}{n}$.

Subcase 2.1: $\alpha \in D(n, k)$ **and** $E(\alpha) \cap A_j = \emptyset$ **for any** $j \in N \setminus \{1\}$

In this subcase, all the objects with disutility α are in A_1, and thus $v(e) < \alpha$ for any $e \in A_j$ and $j \in N \setminus \{1\}$. Due to the normalization, there exists an agent j_0 who receives disutility at most $\frac{1-v(A_1)}{n-1}$, which gives the following lower bound of the difference between the disutilities that agents 1 and j_0 receive

$$v(A_1) - v(A_{j_0}) \geq \frac{n}{n-1} v(A_1) - \frac{1}{n-1} > \frac{1-(k+2)\alpha}{(n-1)(k+1)}.$$

It can be shown that the rightmost-hand side of the above inequality is no less than $\frac{\alpha}{2}$, which is equivalent to $\alpha \leq \frac{2}{(k+1)n+k+3}$. Since $\alpha \leq \frac{k+2}{n(k+1)^2+k+2}$, it suffices to show $\frac{2}{(k+1)n+k+3} \geq \frac{k+2}{n(k+1)^2+k+2}$, which holds since

$$\frac{2}{(k+1)n+k+3} - \frac{k+2}{n(k+1)^2+k+2} = \frac{(k+1)(nk-k-2)}{((k+1)n+k+3)(n(k+1)^2+k+2)} \geq 0,$$

where the last inequality is because $n \geq 3$ and $k \geq 1$, or $n \geq 2$ and $k \geq 2$.

Therefore, $v(A_1) - v(A_{j_0}) > \frac{\alpha}{2}$. Let e^* be an object in A_1 with disutility α. Since $v(A_1) > (k+1)\alpha > \alpha$, for any $S \subseteq A_{j_0}$ with disutility smaller than α, Claim 1 actually gives a tighter bound of its disutility, i.e., $v(S) \leq v(e^*) - (v(A_1) - v(A_{j_0})) < \frac{\alpha}{2}$. Thus, $v(e) < \frac{\alpha}{2}$ for any $e \in A_{j_0}$. Besides, according to Claim 2, $v(A_{j_0}) \geq v(e^*) = \alpha$. These two facts together imply that there exists $S' \subseteq A_{j_0}$ such that $v(S') \in [\frac{\alpha}{2}, \alpha)$, which is a contradiction to Claim 1.

Subcase 2.2: $\alpha \in D(n,k)$ and $E(\alpha) \cap A_{j'} \neq \emptyset$ for some $j' \in N \setminus \{1\}$

In this subcase, some objects with disutility α are in $A_{j'}$. Before diving into the proof for this subcase, we present the following claim, which shows the existence of a subset of A_1 whose disutility is within a specific range.

Claim 3 *There exists a subset $S \subseteq A_1$ such that $\frac{k}{k+2}v(A_1) \leq v(S) < v(A_1) - \alpha$.*

Proof of Claim 3. When $k = 1$, if there exists $e \in A_1$ such that $v(e) \geq \frac{1}{3}v(A_1)$, recall that $v(A_1) > (k+1)\alpha = 2\alpha$, Claim 3 holds since $v(e) \leq \alpha < v(A_1) - \alpha$. If $v(e) < \frac{1}{3}v(A_1)$ for any $e \in A_1$, denote by (A_1^1, A_1^2) one 2-partition of A_1 that minimises the disutility difference between the two bundles among all the 2-partitions. Without loss of generality, we assume $v(A_1^1) \leq v(A_1^2)$, then $v(A_1^1) \leq \frac{1}{2}v(A_1) < v(A_1) - \alpha$. Besides, $v(A_1^1) \geq \frac{1}{3}v(A_1)$ holds. Otherwise, $v(A_1^2) - v(A_1^1) = v(A_1) - 2v(A_1^1) > \frac{1}{3}v(A_1)$, implying that moving an object from A_1^2 to A_1^1 returns another 2-partition of A_1 that has a smaller disutility difference, which contradicts the definition of (A_1^1, A_1^2).

When $k \geq 2$, we first show that $v(e) > \frac{1}{k+2}\alpha$ for any $e \in A_1$. If not, $v(A_1) > v(A_1 \setminus \{e\}) \geq v(A_1) - \frac{1}{k+2}\alpha$. Then Claim 2 gives $v(A_j) \geq v(A_1) - \frac{1}{k+2}\alpha$ for any $j \in N \setminus \{1\}$. Summing up these lower bounds gives the following inequality

$$1 = \sum_{j \in N} v(A_j) \geq v(A_1) + (n-1)v(A_1) - \frac{n-1}{k+2}\alpha > \frac{k+2}{k+1} - \frac{(k+2)^2 + (k+1)(n-1)}{(k+1)(k+2)}\alpha.$$

It can be shown that the rightmost-hand side is at least 1, which constitutes a contradiction. This is equivalent to show that $\alpha \leq \frac{k+2}{(k+2)^2+(k+1)(n-1)}$. Since $\alpha \leq \frac{k+2}{n(k+1)^2+k+2}$, it suffices to show that $\frac{k+2}{(k+1)^2+(k+1)(n-1)} \geq \frac{k+2}{n(k+1)^2+k+2}$, which holds since

$$n(k+1)^2 + k + 2 - ((k+2)^2 + (k+1)(n-1)) = (k+1)(nk - k - 1) \geq 0,$$

where the last inequality is because $n \geq 2$ and $k \geq 1$.

We then let $S^* = \arg\min_{S \subseteq A_1, v(S) > \alpha} v(S)$ which is guaranteed to exist since $v(A_1) > (k+1)\alpha > \alpha$, and show by contradiction that $v(S^*) \leq \frac{2}{k+2}v(A_1)$. This gives $\frac{k}{k+2}v(A_1) \leq v(A_1 \setminus S^*) < v(A_1) - \alpha$. We assume for the sake of contradiction that $v(S^*) > \frac{2}{k+2}v(A_1)$. Then the definition of S^* gives the following lower bound of $v(e)$ for any $e \in S^*$

$$v(e) > v(S^*) - \alpha > \frac{2}{k+2}v(A_1) - \alpha > \frac{k}{k+2}\alpha \geq \frac{1}{2}\alpha,$$

where the second last inequality is because $v(A_1) > (k+1)\alpha$ and the last inequality is because $k \geq 2$. This lower bound implies that S^* contains exactly 2 objects. Otherwise (i.e., $|S^*| \geq 3$), for any subset $S' \subseteq S^*$ that contains exactly 2 objects, $\alpha < v(S') < v(S^*)$ holds, which contradicts the definition of S^*.

Therefore, we can denote $S^* = \{e^l, e^s\}$ and assume without loss of generality that $v(e^l) \geq v(e^s)$. Accordingly, $v(e^l) \geq \frac{1}{2}v(S^*) > \frac{1}{k+2}v(A_1) > \frac{k+1}{k+2}\alpha$. Recall that $v(e) > \frac{1}{k+2}\alpha$ holds for any $e \in A_1$. These two facts together imply that the total disutility of e^l and any other object in A_1 is larger than α. From the definition of S^*, we know that $e^s \in \arg\min_{e \in A_1} v(e)$, which gives $v(e) \geq v(e^s) > \frac{k}{k+2}\alpha$ for any $e \in A_1$. Let S' be the subset of A_1 that contains the two objects with the smallest disutilities, the following inequality leads to a contradiction to the definition of S^*

$$\alpha \le \frac{2k}{k+2}\alpha < v(S') \le \frac{2}{k+2}v(A_1) < v(S^*),$$

where the first inequality is because $k \ge 2$ and the second last inequality is because $|A_1| \ge k+2$. ■

We are now ready to reveal the contradiction in the subcase. Denote by e^* one object in $A_{j'}$ that has disutility α and by S a subset of A_1 that satisfies Claim 3, Claim 2 gives $v(A_{j'} \setminus \{e^*\}) \ge v(S) \ge \frac{k}{k+2}v(A_1)$; that is, $v(A_{j'}) \ge \frac{k}{k+2}v(A_1) + \alpha$. For any $j \in N \setminus \{1, j'\}$, recall that $|A_1| \ge k+2$ which implies that there exists $S' \subseteq A_1$ such that $v(A_1) > v(S') \ge \frac{k+1}{k+2}v(A_1)$, Claim 2 gives $v(A_j) \ge v(S') \ge \frac{k+1}{k+2}v(A_1)$. Summing up these lower bounds leads to the following contradiction

$$\begin{aligned} 1 = \sum_{j\in N} v(A_j) &\ge v(A_1) + \frac{k}{k+2}v(A_1) + \alpha + (n-2)\cdot\frac{k+1}{k+2}v(A_1) \\ &= \frac{n(k+1)}{k+2}v(A_1) + \alpha > 1 - \alpha + \alpha = 1. \end{aligned}$$

For the other direction, the disutility function for the subcases when $\alpha \in D(n,k)$ (see Table 2) containing one object with disutility α and $n(k+1)$ objects with disutility $\frac{1-\alpha}{n(k+1)}$. Since $\alpha > \frac{1}{kn+n+1}$, it follows that $\alpha > \frac{1-\alpha}{n(k+1)}$. Besides, it can be verified that $\alpha < \frac{2-2\alpha}{n(k+1)}$, which is equivalent to $\alpha < \frac{2}{nk+n+2}$. Since $\alpha \le \frac{k+2}{n(k+1)^2+k+2}$, it suffices to show $\frac{k+2}{n(k+1)^2+k+2} < \frac{2}{n(k+1)+2}$, which holds since

$$\frac{2}{n(k+1)+2} - \frac{k+2}{n(k+1)^2+k+2} = \frac{nk(k+1)}{(n(k+1)+2)(n(k+1)^2+k+2)} > 0.$$

By the pigeonhole principle, there exists a bundle that contains at least $k+2$ objects in any allocation. This implies that the MinMaxShare of this disutility function is $(k+2)\cdot\frac{1-\alpha}{n(k+1)}$, which happens in the allocation where one bundle contains $k+2$ objects with disutility $\frac{1-\alpha}{n(k+1)}$, one bundle contains k objects with disutility $\frac{1-\alpha}{n(k+1)}$ and one object with disutility α, and each of the other bundles contains $k+1$ objects with disutility $\frac{1-\alpha}{n(k+1)}$.

Object Disutility	Quantity
α	1
$\frac{1-\alpha}{n(k+1)}$	$n(k+1)$

Table 2: Disutility function for subcases $\alpha \in D(n,k)$ with $n \ge 3$ and $k \ge 1$, or $n \ge 2$ and $k \ge 2$.

Next we consider the subcases when $\alpha \in I(n,k)$. Recall that when $\alpha \in I(n,k)$, $\alpha \in (\frac{k+2}{n(k+1)^2+k+2}, \frac{1}{kn+1}]$ and $v(A_1) > \Delta_n^{\oplus}(\alpha) = (k+1)\alpha$.

Subcase 2.3: $\alpha \in I(n,k)$ and $E(\alpha) \cap A_j = \emptyset$ for any $j \in N \setminus \{1\}$

For this subcase, we first derive a lower bound of $v(A_j)$ for any $j \in N \setminus \{1\}$, i.e., $v(A_j) \ge (\frac{(k+1)^2}{k+2} + \frac{1}{(n-1)(k+2)})\alpha$. Let $D = \frac{(k+1)^2}{k+2} + \frac{1}{(n-1)(k+2)}$, we assume for the sake of contradiction that $v(A_{j'}) < D\alpha$ for some $j' \in N \setminus \{1\}$. It can be verified that $k < D < k+1$, where the first inequality is equivalent to $n > 0$, and the second inequality is equivalent to $(n-1)(k+1) > 1$. Denote by e^* one object in A_1 with disutility α and by e' any object in A_j, we have

$$v(A_{j'} \setminus (A_{j'} \setminus \{e'\}) \cup (A_1 \setminus \{e^*\})) = v(A_1 \setminus \{e^*\} \cup \{e'\}) < v(A_1).$$

Then from Claim 2, $v(A_{j'} \setminus \{e'\}) \geq v(A_1 \setminus \{e^*\})$, which gives

$$v(e') \leq v(A_{j'}) - v(A_1) + v(e^*) < D\alpha - (k+1)\alpha + \alpha = (D-k)\alpha.$$

However, we next show that the disutility of some object in $A_{j'}$ must be larger than $(D-k)\alpha$, which leads to a contradiction. To achieve this, we denote $S^* \in \arg\min_{S \subseteq A_{j'}, v(S) > (D-1)\alpha} v(S)$, whose existence is guaranteed since Claim 2 gives $v(A_{j'}) \geq v(A_1 \setminus \{e^*\}) > k\alpha > (D-1)\alpha$. Notice that

$$v(A_{j'} \setminus S^* \cup (A_1 \setminus \{e^*\})) < D\alpha - (D-1)\alpha + v(A_1) - \alpha = v(A_1),$$

from Claim 2, $v(S^*) \geq v(A_1 \setminus \{e^*\}) > k\alpha$. Then the definition of S^* implies that the disutility of any object in S^* is at least

$$v(S^*) - (D-1)\alpha > (k - D + 1)\alpha \geq (D-k)\alpha,$$

where the last inequality is equivalent to $D - k - \frac{1}{2} = \frac{k+2-kn}{2(n-1)(k+2)} \leq 0$, which holds when $n \geq 3$ and $k \geq 1$, or $n \geq 2$ and $k \geq 2$.

Therefore, $v(A_j) \geq (\frac{(k+1)^2}{k+2} + \frac{1}{(n-1)(k+2)})\alpha$ holds for any $j \in N \setminus \{1\}$. Summing up these lower bounds leads to the following contradiction

$$1 = \sum_{j \in N} v(A_j) > (k+1)\alpha + (n-1) \cdot (\frac{(k+1)^2}{k+2} + \frac{1}{(n-1)(k+2)})\alpha = \frac{n(k+1)^2 + k + 2}{k+2}\alpha > 1.$$

Subcase 2.4: $\alpha \in I(n,k)$ and $E(\alpha) \cap A_{j'} \neq \emptyset$ for some $j' \in N \setminus \{1\}$

The proof is similar to that of Subcase 2.2. First, it can be verified that Claim 3 still holds.
Proof of Claim 3 for $\alpha \in I(n,k)$. Notice that Claim 3 holds as long as $k = 1$, thus, we can focus on $k \geq 2$. We first show that $v(e) > \frac{1}{k+2}\alpha$ for any $e \in A_1$. If not, $v(A_1 \setminus \{e\}) \geq v(A_1) - \frac{1}{k+2}\alpha$. Then Claim 2 gives $v(A_j) \geq v(A_1) - \frac{1}{k+2}\alpha$ for any $j \in N \setminus \{1\}$. Summing up these lower bounds gives the following formula

$$1 = \sum_{j \in N} v(A_j) \geq v(A_1) + (n-1)v(A_1) - \frac{n-1}{k+2}\alpha > \frac{n(k+1)(k+2) - n + 1}{k+2}\alpha.$$

It can be shown that the rightmost-hand side of the above inequality is at least 1, which is a contradiction. This is equivalent to show that $\alpha \geq \frac{k+2}{n(k+1)(k+2)-n+1}$. Since $\alpha \geq \frac{k+2}{n(k+1)^2+k+2}$, it suffices to show that $\frac{k+2}{n(k+1)(k+2)-n+1} \leq \frac{k+2}{n(k+1)^2+k+2}$, which holds since

$$n(k+1)(k+2) - n + 1 - (n(k+1)^2 + k + 2) = nk - k - 1 \geq 0$$

where the last inequality is because $n \geq 2$ and $k \geq 1$.

We then let $S^* = \arg\min_{S \subseteq A_1, v(S) > \alpha} v(S)$, which is guaranteed to exist since $v(A_1) > (k+1)\alpha > \alpha$. By the same proof as the counterpart in the proof of Claim 3 for $\alpha \in D(n,k)$, we can show that $v(S^*) \leq \frac{2}{k+2}v(A_1)$, which gives $\frac{k}{k+2}v(A_1) \leq v(A_1 \setminus S^*) < v(A_1) - \alpha$. ■

We are now ready to reveal the contradiction in this subcase. Denote by e^* one object in $A_{j'}$ that has disutility α and by S a subset of A_1 that satisfies Claim 3, Claim 2 gives $v(A_{j'} \setminus \{e^*\}) \geq v(S) \geq \frac{k}{k+2}v(A_1)$; that is, $v(A_{j'}) \geq \frac{k}{k+2}v(A_1) + \alpha$. For any $j \in N \setminus \{1, j'\}$, recall that $|A_1| \geq k+2$ which implies that there exists $S' \subseteq A_1$ such that $v(A_1) > v(S') \geq \frac{k+1}{k+2}v(A_1)$, Claim 2 gives $v(A_j) \geq v(S') \geq \frac{k+1}{k+2}v(A_1)$. Summing up these lower bounds leads to the following contradiction

$$\begin{aligned} 1 = \sum_{j \in N} v(A_j) &\geq v(A_1) + \frac{k}{k+2}v(A_1) + \alpha + (n-2) \cdot \frac{k+1}{k+2}v(A_1) \\ &= \frac{n(k+1)}{k+2}v(A_1) + \alpha > \frac{n(k+1)^2 + k + 2}{k+2}\alpha > 1. \end{aligned}$$

For the other direction, the disutility function for the subcases when $\alpha \in I(n,k)$ (See Table 3) containing $kn+1$ objects with disutility α and $n-1$ objects with disutility $\frac{1-(nk+1)\alpha}{n-1}$. It can be verified that $\alpha > \frac{1-(kn+1)\alpha}{n-1}$, which is equivalent to $\alpha > \frac{1}{(k+1)n}$. Since $\alpha > \frac{k+2}{n(k+1)^2+k+2}$, it suffices to show $\frac{k+2}{n(k+1)^2+k+2} \geq \frac{1}{(k+1)n}$, which holds since

$$\frac{k+2}{n(k+1)^2+k+2} - \frac{1}{(k+1)n} = \frac{(k+1)n-k-2}{(n(k+1)^2+k+2)(k+1)n} \geq 0,$$

where the inequality is because $n \geq 3$ and $k \geq 1$, or $n \geq 2$ and $k \geq 2$. By the pigeonhole principle, there exists a bundle that contains at least $k+1$ objects with disutility α. This implies that the MinMaxShare of this disutility function is $(k+1)\alpha$, which happens in the allocation where one bundle contains $k+1$ objects with disutility α, and each of the other bundles contains k objects with disutility α and one object with disutility $\frac{1-(nk+1)\alpha}{n-1}$.

Object Disutility	Quantity
α	$kn+1$
$\frac{1-(nk+1)\alpha}{n-1}$	$n-1$

Table 3: Disutility function for subcases $\alpha \in I(n,k)$ with $n \geq 3$ and $k \geq 1$, or $n \geq 2$ and $k \geq 2$.

Up to here, we computed Hill's share for unrestricted m, except for $n=2$ and $k=1$. Moving to the setting of restricted m, Hill's share can be computed by similar approaches with a more involved discussion. The remaining proof of Theorem 1 is regulated to the appendix.

4 Hill's Guarantee for Indivisible Bads

4.1 Main Result

In this section, we prove the counterpart result of Hill's guarantee for indivisible bads. Consider the general case, where each one of the n agents now has an arbitrary disutility v_i in $\mathcal{A}dd(M)$ (by convention $m = |M|$). Given m and n, a *guarantee* specifies an upper bound $\Gamma_n(v_i; m)$ on agent i's disutility when she shares the m bads with $n-1$ other agents of unknown disutilities in $\mathcal{A}dd(M)$. By construction the mapping Γ_n is the same for every agent i. As part of its definition, a guarantee must be feasible: for any profile $(v_i)_{i=1}^n \in [\mathcal{A}dd(M)]^n$ there exists an allocation $(A_1, \ldots, A_n)$ of M such that

$$v_i(A_i) \leq \Gamma_n(v_i; m) \text{ for all } 1 \leq i \leq n. \tag{4}$$

We know from Aziz et al. (2017) and Feige et al. (2021) that the MinMaxShare value $\mathsf{MMS}_n(v_i)$ is not a guarantee because at some (rare!) profiles no allocation meets all inequalities in (4). By applying Inequalities (4) to an arbitrary guarantee Γ_n at the unanimous profile $v_i = v$ for all i, we see it is lower bounded by the MinMaxShare:

$$\Gamma_n(v; m) \geq \mathsf{MMS}_n(v) \text{ for all } v \in \mathcal{A}dd(M).$$

In this section, we show that the monotone hull of $\Delta_n^{\oplus}$ serves as the best guarantee in Hill's model. Recall that $\mathcal{U}(\alpha; m)$ contains all the disutility functions $v(\cdot)$ on objects M such that $\max_{e \in M} v(e) \leq \alpha$, and $\mathcal{U}(\alpha) = \bigcup_m \mathcal{U}(\alpha; m)$. For simplicity in the presentation and analysis, we ignore the restriction of the number of objects m in this section, and the result can be extended to the setting with parameter m using the same approach in Section 3. The definition of $\mathcal{U}(\alpha)$ is the same as in (Hill, 1987; Markakis and Psomas, 2011; Gourvès et al., 2015). Note that $\mathcal{U}(\alpha') \subseteq \mathcal{U}(\alpha)$ if $\alpha' \leq \alpha$, and the difference between $\mathcal{V}(\alpha)$ and $\mathcal{U}(\alpha)$ is that the disutilities

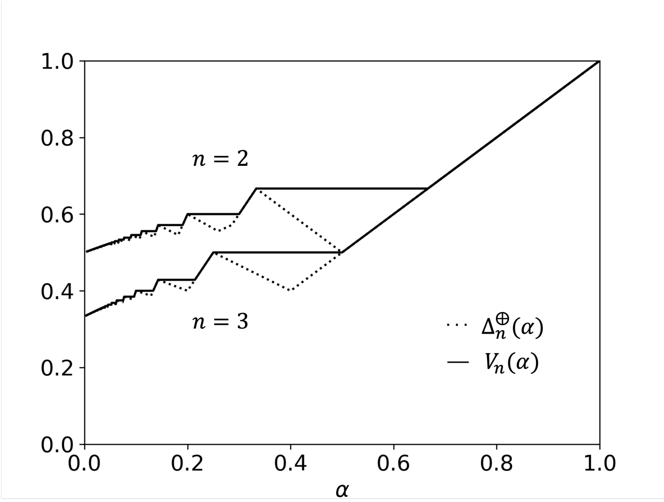


Figure 3: The characterisation for Heterogeneous Agents

in $\mathcal{U}(\alpha)$ do not require that there must be one object with disutility α. It is straightforward that the tight guarantee regarding $\mathcal{U}(\cdot)$ must be monotone non-decreasing since any worst-case disutility in $\mathcal{U}(\beta)$ is also a disutility in $\mathcal{U}(\alpha)$ for $\beta \leq \alpha$. We write V_n the monotone hull of $\Delta_n^{\oplus}$

$$V_n(\alpha) = \max_{0 \leq \beta \leq \alpha} \Delta_n^{\oplus}(\beta),$$

as illustrated in Fig. 3 when $n = 2, 3$. In more detail, we have the following formula of V_n:

$$V_n(\alpha) = \begin{cases} \frac{k+2}{(k+1)n+1}, & \text{if } \alpha \in NI(n,k) \\ (k+1)\alpha, & \text{if } \alpha \in I(n,k) \end{cases}$$

where for any integer $k \geq 0$,

$$NI(n,k) = \left(\frac{1}{(k+1)n+1}, \frac{k+2}{(k+1)((k+1)n+1)} \right)$$

and

$$I(n,k) = \left[\frac{k+2}{(k+1)((k+1)n+1)}, \frac{1}{kn+1} \right].$$

By Theorem 1 and the construction of $V_n(\cdot)$, $V_n(\cdot)$ provides the tight bound of the worst-case MinMaxShare regarding $\mathcal{U}(\cdot)$. We further prove that $V_n(\cdot)$ is a guarantee and moreover an allocation satisfying $V_n(\cdot)$ can be found in polynomial time.

Theorem 2 $\Gamma_n(v) = V_n(\max_{e \in M} v(e))$ *defines a canonical guarantee. That is, given any* $0 \leq \alpha_i \leq 1$ *and* $v_i \in \mathcal{U}(\alpha_i)$ *for* $i = 1, \ldots, n$, *there exists an allocation* $(A_1, \ldots, A_n)$ *with*

$$v_i(A_i) \leq V_n(\alpha_i) \text{ for all } i = 1, \ldots, n$$

and such an allocation can be computed in polynomial time. Moreover, for any $0 \leq \alpha \leq 1$, *there exists* $\{v'_i\}_{i=1}^n$ *with* $v'_i \in \mathcal{U}(\alpha)$ *for any* $i \in [n]$ *such that* $V_n(\alpha)$ *is the best possible guarantee, i.e., for any allocation* $(B_1, \ldots, B_n)$,

$$\text{there exists } i \in N \text{ such that } v'_i(B_i) \geq V_n(\alpha).$$

As for $\Delta_n^{\oplus}(\cdot)$ in Theorem 1 the two key features of this guarantee are: its computation is elementary and it does not depend on the number of bads to allocate. As far as we know, no other similarly simple guarantee for the allocation of bads has been identified.

Remark 2 *By Theorem 2, $V_n(\alpha)$ is the best guarantee for disutilities in $\mathcal{U}(\alpha)$, and thus we get the tight counterpart result of Hill (1987) for bads. However, it may not be the best in the model of Gourvès et al. (2015), i.e., for disutilities in $\mathcal{V}(\alpha)$. For example, when $n = 2$, we can show that $\Delta_2^{\oplus}(\max_{e\in M} v_i(e))$ is a tighter guarantee in the later model. Given two disutility functions v_1 and v_2, without loss of generality, suppose $\Delta_2^{\oplus}(\max_e v_1(e)) \leq \Delta_2^{\oplus}(\max_e v_2(e))$. Then we find the MinMax partition of v_1 so that the disutilities of both bundles are no greater than $\Delta_2^{\oplus}(\max_e v_1(e))$ to agent 1. We ask agent 2 to choose a better bundle whose disutility must be no greater than $\frac{1}{2}$ and thus no greater than $\Delta_2^{\oplus}(\max_e v_2(e))$ to agent 2. It is still open whether $\Delta_n^{\oplus}(\max_{e\in M} v_i(e))$ is a guarantee or not when $n \geq 3$ in Gourvès et al. (2015)'s model, which is an interesting future research direction.*

4.2 Proof of Theorem 2

To show that one can compute an allocation satisfying the required bound in Theorem 2, we derive a variation of the moving-knife algorithm. When the objects are goods and divisible, Dubins and Spanier (1961) proved that such an algorithm (also known as Dubins-Spanier moving knife algorithm) gives the optimal worst-case bound, i.e., every agent gets value for at least $\frac{1}{n}$. Markakis and Psomas (2011) further proved that a variation of this algorithm also guarantees the optimal worst-case bound for indivisible goods. In a nutshell, towards proving Theorem 2, we first use the reduction proved in (Bouveret and Lemaître, 2016; Huang and Lu, 2021) to restrict our attention to the ordered instances when agents have the same ranking over all objects, which significantly simplifies our analysis. Then we show that using $V_n(\cdot)$ to set the parameters in the moving-knife algorithm always returns an allocation ensuring the bound in Theorem 2.

The following lemma says that it suffices to only focus on the ordered instances.

Lemma 3 ((Bouveret and Lemaître, 2016; Huang and Lu, 2021)) *Suppose there is an algorithm that takes any ordered instance as input, runs in $T(n, m)$ time and returns an allocation where each agent i's disutility is at most $V_n(\alpha_i)$. Then, we have an algorithm that takes any instance as input, runs in $T(n, m) + O(nm \log m)$ time and returns an allocation with the same disutility guarantee.*

Our approach is similar to that in Markakis and Psomas (2011), but the detailed proof differs. Our algorithm runs in recursions. In each recursion, the algorithm allocates a bundle of objects to one agent in a moving-knife fashion. Each time, each of the remaining agents moves her "knife" one object towards the objects with smaller disutilities, until for every agent i the total disutility of the objects before her "knife" is larger than $V_n(\alpha_i)$. After that, one of the last agents (denoted by agent k) for whom the total utility of the objects before her knife is larger than $V_n(\alpha_k)$ receives the objects except the one right before her knife. If there remains only one agent who has not received a bundle, she will get all the remaining objects. Otherwise, all the remaining agents enter the next recursion with their disutility functions being normalized such that for each of them the total disutility of the remaining objects is 1. The formal description of our algorithm is presented in Algorithm 1.

Then we are going to prove Theorem 2. Without loss of generality, let $1, \ldots, n$ be the order in which agents receive bundles in Algorithm 1. Denote $C_i = v_i(A_1)$ for every $N \setminus \{1\}$, the following lemma gives a lower bound of C_i.

Lemma 4 *For any agent $i \in N \setminus \{1\}$ with $\alpha_i \in NI(n, k) \cup I(n, k)$ for some $k \geq 0$, we have*

$$C_i \geq \frac{1 - V_n(\alpha_i)}{n - 1}.$$

Proof. Denote by q the index such that $\sum_{e=1}^{q} v_i(e) \leq V_n(\alpha_i)$ and $\sum_{e=1}^{q+1} v_i(e) > V_n(\alpha_i)$, whose existence is guaranteed since $v_i(M) > V_n(\alpha_i)$. Since $V_n(\alpha_i) \geq (k + 1)\alpha_i$ (this can be easily

Algorithm 1 Algorithm for heterogeneous disutilities

Input: An ordered instance with agents N, objects M and disutility functions $\{v_i\}_{i \in N}$.

Output: An allocation $\mathbf{A} = \{A_1, \ldots, A_n\}$ with $v_i(A_i) \leq V_n(\alpha_i)$ for every $i \in N$.

```
Initialize S_i = ∅ for every i ∈ N.
while there exists an agent j with v_j(S_j) ≤ V_n(α_j) do
   for every i ∈ N do
      S_i ← S_i ∪ {the object in M \ S_i with the largest disutility for agent i (tie breaks arbitrarily)}.
   end for
end while
Pick the agent k ∈ N with v_k(S_k \ {ẽ}) ≤ V_n(α_k) where ẽ is the last object that k added
   into S_k (tie breaks arbitrarily).
A_k ← S_k \ {ẽ}.
if |N| = 2 then
   Allocate M \ A_k to the remaining agent.
else
   Construct a new disutility function v'_i for every i ∈ N \ {k} by setting v'_i(e) = v_i(e)/(1 − v_i(A_k)) for
      every e ∈ M \ A_k.
   Run Algorithm 1(N \ {k}, M \ A_k, {v'_i}_{i∈N\{k}}).
end if
```

verified from the definition of $V_n(\alpha)$ and can also be seen from Fig. 3) and $v_i(e) \leq \alpha_i$ for every $e \in M$, $q \geq k+1$. Otherwise, $\sum_{e=1}^{q+1} v_i(e) \leq (k+1)\alpha_i \leq V_n(\alpha_i)$, which contradicts the definition of q. According to Algorithm 1, $C_i \geq \sum_{e=1}^{q} v_i(e)$. Since only ordered instances are considered, $v_i(\{q+1\}) \leq v_i(\{q\}) \leq \frac{C_i}{k+1}$, which gives

$$C_i + \frac{C_i}{k+1} \geq \sum_{e=1}^{q+1} v_i(e) > V_n(\alpha_i).$$

Therefore, $C_i > \frac{k+1}{k+2} \cdot V_n(\alpha_i)$. We consider the following two cases regarding the ranges of α_i.

Case 1: $\alpha_i \in I(n,k)$. In this case, $\frac{k+2}{(k+1)((k+1)n+1)} \leq \alpha_i \leq \frac{1}{kn+1}$ and $V_n(\alpha_i) = (k+1)\alpha_i$. Then,

$$C_i > \frac{k+1}{k+2} \cdot V_n(\alpha) \geq \frac{1 - V_n(\alpha)}{n-1},$$

where the last inequality holds since $\alpha_i \geq \frac{k+2}{(k+1)((k+1)n+1)}$.

Case 2: $\alpha_i \in NI(n,k)$. In this case, $V_n(\alpha_i) = \frac{k+2}{(k+1)n+1}$, which gives

$$C_i > \frac{k+1}{k+2} \cdot V_n(\alpha) = \frac{1 - V_n(\alpha)}{n-1},$$

which completes the proof. ■

Interestingly, the following lemma shows the connection between the ranges of α_i and $\frac{\alpha_i}{1 - \frac{1 - V_n(\alpha_i)}{n-1}}$.

Lemma 5 *For any $\alpha_i \in NI(n,k) \cup I(n,k)$ for some $k \geq 0$, we have*

$$\frac{\alpha_i}{1 - \frac{1 - V_n(\alpha_i)}{n-1}} \in \begin{cases} I(n-1,k), & \text{if } \alpha_i \in I(n,k) \\ NI(n-1,k), & \text{if } \alpha_i \in NI(n,k) \end{cases}$$

Proof. We consider the following two cases regarding the ranges of α_i.

Case 1: $\alpha_i \in I(n,k)$. In this case, $\frac{k+2}{(k+1)((k+1)n+1)} \leq \alpha_i \leq \frac{1}{kn+1}$ and $V_n(\alpha_i) = (k+1)\alpha_i$. Then, we have

$$\frac{\alpha_i}{1 - \frac{1-V_n(\alpha_i)}{n-1}} = \frac{(n-1)\alpha_i}{n-2+(k+1)\alpha_i} \leq \frac{n-1}{(n-2)(kn+1)+k+1} = \frac{1}{k(n-1)+1},$$

where the inequality is because $\alpha_i \leq \frac{1}{kn+1}$. Besides,

$$\frac{\alpha_i}{1 - \frac{1-V_n(\alpha_i)}{n-1}} = \frac{(n-1)\alpha_i}{n-2+(k+1)\alpha_i} \geq \frac{(k+2)(n-1)}{(k+1)((n-2)(kn+n+1)+k+2)}$$
$$= \frac{k+2}{(k+1)((k+1)(n-1)+1)},$$

where the inequality is because $\alpha_i \geq \frac{k+2}{(k+1)((k+1)n+1)}$.

Case 2: $\alpha_i \in NI(n,k)$. In this case, $\frac{1}{(k+1)n+1} < \alpha_i < \frac{k+2}{(k+1)((k+1)n+1)}$ and $V_n(\alpha_i) = \frac{k+2}{(k+1)n+1}$. Then, we have

$$\frac{\alpha_i}{1 - \frac{1-V_n(\alpha_i)}{n-1}} = \frac{((k+1)n+1)\alpha_i}{(k+1)(n-1)+1} < \frac{k+2}{(k+1)((k+1)(n-1)+1)},$$

where the inequality is because $\alpha_i < \frac{k+2}{(k+1)((k+1)n+1)}$. Besides,

$$\frac{\alpha_i}{1 - \frac{1-V_n(\alpha_i)}{n-1}} = \frac{((k+1)n+1)\alpha_i}{(k+1)(n-1)+1} > \frac{1}{(k+1)(n-1)+1},$$

where the inequality is because $\alpha_i > \frac{1}{(k+1)n+1}$. ■

Proof of Theorem 2. We prove Theorem 2 by mathematical induction. When $n = 2$, it is easy to see the correctness of Theorem 2 from Lemma 4 since $v_1(A_1) \leq V_2(\alpha_1)$ and $v_2(A_2) = 1 - v_2(A_1) \leq 1 - (1 - V_2(\alpha_2)) = V_2(\alpha_2)$, We assume as our induction hypothesis that Theorem 2 holds for $n-1$. Then we aim to prove the correctness for n.

From Algorithm 1, $v_1(A_1) \leq V_n(\alpha_1)$ clearly holds for agent 1. For any other agent $i \in N\backslash\{1\}$, denote $\widetilde{\alpha}_i = \max_{e \in M\backslash A_1} v'_i(e)$. We know from Algorithm 1 that $\widetilde{\alpha}_i \leq \frac{\alpha_i}{1-C_i}$ and from the induction hypothesis that $v'_i(A_i) \leq V_{n-1}(\widetilde{\alpha}_i)$, which together give

$$v_i(A_i) = (1-C_i)v'_i(A_i) \leq (1-C_i)V_{n-1}(\widetilde{\alpha}_i) \leq (1-C_i)V_{n-1}(\frac{\alpha_i}{1-C_i}),$$

where the last inequality holds by recalling that $V_{n-1}(\widetilde{\alpha}_i)$ is an non-decreasing function of $\widetilde{\alpha}_i$. Therefore, it remains to show

$$(1-C_i)V_{n-1}(\frac{\alpha_i}{1-C_i}) \leq V_n(\alpha_i).$$

Note that $(1-C_i)V_{n-1}(\frac{\alpha_i}{1-C_i})$ is an non-increasing function of C_i. This is because when $\frac{\alpha_i}{1-C_i} \in I(n-1,k)$ for some k, $(1-C_i)V_{n-1}(\frac{\alpha_i}{1-C_i}) = (1-C_i)(k+1)\frac{\alpha_i}{1-C_i} = (k+1)\alpha_i$, which is a constant with respect to C_i; when $\frac{\alpha_i}{1-C_i} \in NI(n-1,k)$ for some k, $(1-C_i)V_{n-1}(\frac{\alpha_i}{1-C_i}) = (1-C_i)\frac{k+2}{(k+1)(n-1)+1}$, a decreasing function of C_i. Then, the following formula completes the proof of Theorem 2

$$(1-C_i)V_{n-1}(\frac{\alpha_i}{1-C_i}) \leq (1 - \frac{1-V_n(\alpha_i)}{n-1})V_{n-1}(\frac{\alpha_i}{1 - \frac{1-V_n(\alpha_i)}{n-1}}),$$

where the first inequality is due to $C_i \geq \frac{1-V_n(\alpha_i)}{n-1}$ (according to Lemma 4), and the second inequality can be verified by considering the following two cases regarding the ranges of α_i,

Case 1: $\alpha_i \in I(n,k)$. In this case, Lemma 5 gives $\frac{\alpha_i}{1-\frac{1-V_n(\alpha_i)}{n-1}} \in I(n-1,k)$. Thus, we have

$$\begin{aligned}(1-\frac{1-V_n(\alpha_i)}{n-1})V_{n-1}(\frac{\alpha_i}{1-\frac{1-V_n(\alpha_i)}{n-1}}) &= (1-\frac{1-V_n(\alpha_i)}{n-1})\cdot(k+1)\frac{\alpha_i}{1-\frac{1-V_n(\alpha_i)}{n-1}}\\ &= (k+1)\alpha_i = V_n(\alpha_i).\end{aligned}$$

Case 2: $\alpha_i \in NI(n,k)$. In this case, $\frac{\alpha_i}{1-\frac{1-V_n(\alpha_i)}{n-1}} \in NI(n-1,k)$. Thus, we have

$$\begin{aligned}(1-\frac{1-V_n(\alpha_i)}{n-1})V_{n-1}(\frac{\alpha_i}{1-\frac{1-V_n(\alpha_i)}{n-1}}) &= (1-\frac{1-\frac{k+2}{(k+1)n+1}}{n-1})\cdot\frac{k+2}{(k+1)(n-1)+1}\\ &= \frac{k+2}{(k+1)n+1} = V_n(\alpha_i).\end{aligned}$$

Therefore, we complete the proof of Theorem 2. ∎

The instances provided in Section 3 and Appendix B show the tightness of Theorem 2.

5 Numerical Experiments

To demonstrate that $\Delta_n^{\oplus}(\alpha;m)$ can serve as a good alternative of MinMaxShare, we first evaluate the worst-case ratio of $\Delta_n^{\oplus}(\alpha;m)$ and $\Delta_n^{\ominus}(\alpha;m)$ (recall that $\Delta_n^{\ominus}(\alpha;m)$ is the best-case MinMaxShare over all disutilities in $\mathcal{V}(\alpha;m)$). Denote by $r_n(\alpha;m)=\frac{\Delta_n^{\oplus}(\alpha;m)}{\Delta_n^{\ominus}(\alpha;m)}$. It is clear that $r_n(\alpha;m)$ is no smaller than the ratio between $\Delta_n^{\oplus}(\alpha;m)$ and the real MinMaxShare, and we have illustrated $r_n(\alpha;\infty)$ in Fig. 2 for $n=2,10,100$. As we can see, although the worst-case ratio can be close to 2, it only happens for sufficiently large n and a small range of values of α. For any n and most values of α, the ratio is better than $\frac{4}{3}$ and $\frac{11}{9}$, which are two fractions of the MinMaxShare that are known to be achievable. Actually, it is not hard to verify that $r_n(\alpha;m)\le\frac{2n}{n+1}<2$ for all α, and $r_n(\alpha;m)\le\frac{4}{3}$ for all α outside of $(\frac{4}{9n},\frac{3}{2n+3})$; we provide simple proofs in the appendix. Note that $\frac{3}{2n+3}-\frac{4}{9n}<\frac{7}{6n}$.

Claim 4 *For any $n\ge2$, $\alpha\in(0,1]$ and $m\ge\lceil\frac{1}{\alpha}\rceil$, $r_n(\alpha;m)\le\frac{2n}{n+1}$.*

Claim 5 *$r_n(\alpha;m)>\frac{4}{3}$ only when $\alpha\in(\frac{2}{9},\frac{1}{3})$ if $n=3$, or $\alpha\in(\frac{1}{6},\frac{3}{11})$ if $n=4$, or $\alpha\in(\frac{4}{45},\frac{1}{9})\cup(\frac{2}{15},\frac{3}{13})$ if $n=5$, or $\alpha\in(\frac{4}{9n},\frac{3}{2n+3})$ if $n\ge6$.*

From the formula of $r_n(\alpha;m)$, as well as Fig. 2, we have the following observations:

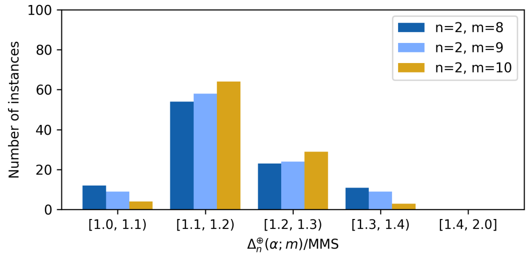
Observation 1 As n increases, the worst-case ratio of $r_n(\alpha;m)$, i.e., $\max_\alpha r_n(\alpha;m)$, increases.

Observation 2 As n increases, large values of $r_n(\alpha;m)$ happen increasingly more rarely if α is randomly generated from $[0,1]$.

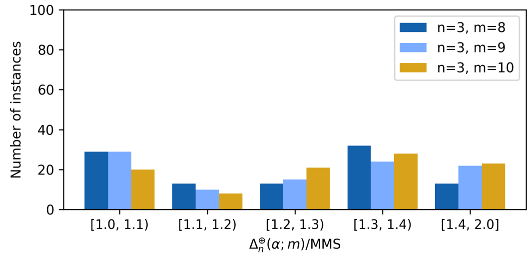
Next, we conduct numerical experiments with synthetic and real-world data to illustrate the real distances between $\Delta_n^{\oplus}(\alpha;m)$ and the MinMaxShare of specific disutility functions, which also validate the above two observations.

5.1 Experiments with Synthetic Data

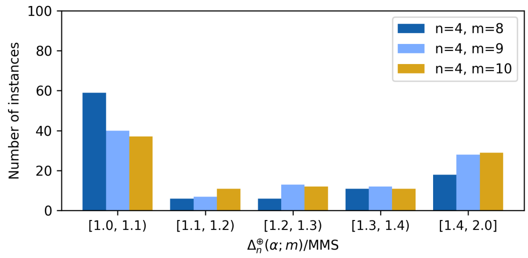
In this section, we randomly generate a number of disutility functions, and for each of them, we compute the ratio between the corresponding Hill's share and the MinMaxShare. In particular, for each given n and m, we randomly generate 100 instances; for each instance, we randomly generate $m-1$ numbers in $[0,1]$. These $m-1$ numbers separate the interval $[0,1]$ into m



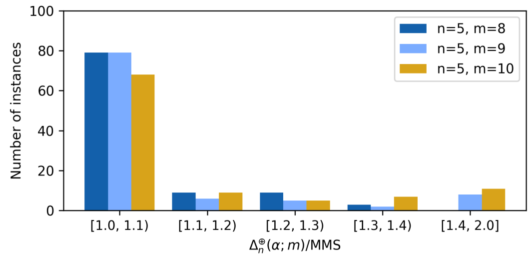
(a) $n = 2$, $m = 8, 9, 10$



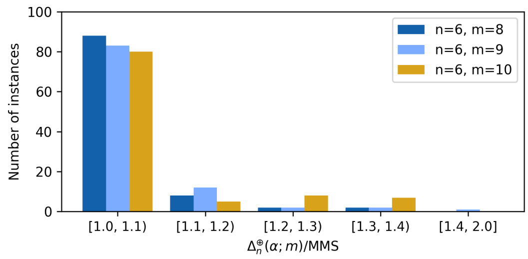
(b) $n = 3$, $m = 8, 9, 10$



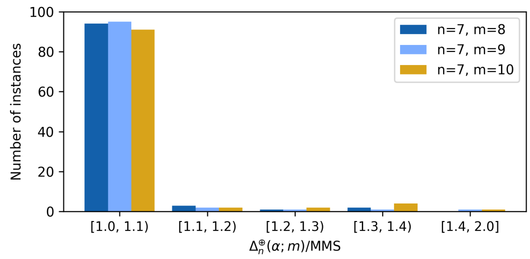
(c) $n = 4$, $m = 8, 9, 10$



(d) $n = 5$, $m = 8, 9, 10$



(e) $n = 6$, $m = 8, 9, 10$



(f) $n = 7$, $m = 8, 9, 10$

Figure 4: Ratios in random data.

contiguous segments, and the lengths of these segments are used as the disutilities of the m objects. Then we compute the $\Delta_n^{\oplus}(\alpha; m)$ value using the maximum of these values and the MinMaxShare. For each instance, we record the ratio of these two quantities.

The results are summarized in Fig. 4. We slice the ratios into small intervals, each of which has a length of 0.1, and count the number of instances falling into each interval for each setting. The figure validates the previous two observations: when $n = 2$ and 3, the largest ratio can only reach interval $[1.3, 1.4)$ and $[1.4, 1.5)$, but when $n \geq 4$, it reaches $[1.5, 1.6)$; however, looking at the number of instances, for larger n, fewer and fewer instances fall into these large intervals, and instead, the number of instances in $[1.0, 1.1)$ significantly dominates the other intervals. Specifically, when $n = 6$ and 7, $[1.0, 1.1)$ contains over 80% of all random instances, and none of them reaches a ratio beyond 1.6, while the worst-case ratio can be greater than 1.7.

In the appendix, we conduct more experiments by fixing $n = 2$ and increasing the value of m and report the change in the distribution of the ratios.

5.2 Experiments with Real-World Data

The real-world data set is collected from the Spliddit platform (spliddit.org) – a well-known platform that provides implementations of fair allocation algorithms for various practical problems (Goldman and Procaccia, 2014). The data set contains 8,409 instances created between October 2014 and May 2020, involving 22,530 agents and 42,469 objects. We randomly select

10,000 disutility functions from the data, where the largest value of n is 14. After normalising all the disutility functions, for each of them, we record the ratio of the corresponding Hill's share and the MinMaxShare. The results are shown in Fig. 5. As we can see, very few instances have ratios higher than 1.4, and over 65% of the instances have ratios within $[1.0, 1.1)$. Actually, there are only 173 ($= 1.73\%$) and 26 ($= 0.26\%$) instances falling into $[1.6, 1.7)$ and $[1.7, 1.8)$ respectively, and none is beyond 1.8. Note that in the 10,000 disutility functions, there are only 14 instances with $n \geq 9$, which further amplifies the rare happening of large ratios.

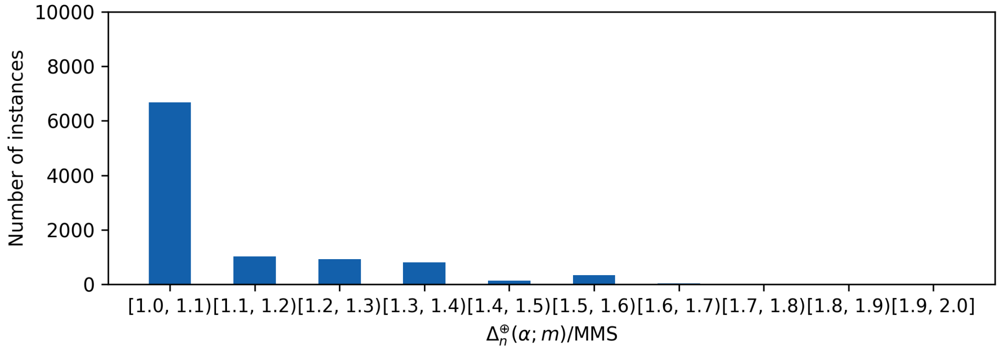


Figure 5: Ratios in Spliddit data.

6 Conclusion

In this work, we give the tight characterisation of Hill's share for allocating indivisible bads, i.e., the exact upper bound of the MinMaxShare of disutility functions with the same largest single-object value. Hill's share exhibits several advantages including elementary computation, being close to the MinMaxShare, and displaying the effect of an agent's disutility in her share of all objects. More importantly, the monotonic cover of Hill's share serves as a canonical guarantee; as far as we know, no other similarly simple guarantee for the allocation of bads has been identified. There are some open problems. Hill's guarantee is tight for the domain of disutility functions whose largest single-object disutility is *no greater than* a given parameter, but we do not know whether it is tight when the domain only contains the disutility functions whose largest single-object disutility *equals* this parameter. The same problem is also open for the mirror problem of allocating goods, for which the tight characterisation of Hill's share is also unknown (when $n \geq 3$). Our work also uncovers some other related research problems, such as the algorithmic problem of finding a Pareto optimal allocation satisfying Hill's share and the game-theoretic problem of designing truthful mechanisms to incentivize the agents to report their disutility functions honestly while achieving (approximations of) Hill's share.

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Appendix

A Missing Proofs in Section 2

A.1 Proof of Lemma 1

For each case, we show that $\mathsf{MMS}_n(v) \geq \Delta_n^{\ominus}(\alpha; m)$ for any $v \in \mathcal{V}(\alpha; m)$, and design a disutility function such that the MinMaxShare is exactly $\Delta_n^{\ominus}(\alpha; m)$. By the definition of $\mathcal{V}(\alpha; m)$, there exists an object with disutility α, thus $\mathsf{MMS}_n(v) \geq \alpha$ for any $v \in \mathcal{V}(\alpha; m)$. Moreover, when $\alpha > 1/n$, there exists a disutility function such that the MinMaxShare is exactly α. Specifically, v_1 contains $\lceil \frac{1}{\alpha} \rceil$ objects, $\lfloor \frac{1}{\alpha} \rfloor$ with disutility α and one with disutility $(1 - \lfloor \frac{1}{\alpha} \rfloor \cdot \alpha) < \alpha$ (if 1 is indivisible by α). $\mathsf{MMS}_n(v_1) = \alpha$ follows from the fact that v_1 contains at most n objects.

By the definition of MinMaxShare, $\mathsf{MMS}_n(v) \geq \frac{1}{n}$, where the equality is achieved when the total disutility of M can be evenly distributed among the n-partition. When $1/n$ is divisible by α (i.e., $\alpha = \frac{1}{kn}$ for some positive integer k), or $1/n$ is not divisible by α (i.e., $\frac{1}{(k+1)n} < \alpha < \frac{1}{kn}$) and the number of objects m is at least $kn + n$, there exists an disutility function such that the MinMaxShare is exactly $1/n$. For the former, the disutility function v_2 contains $1/\alpha = kn$ objects with disutility α. Clearly, each bundle in the best n-partition contains k objects with disutility α and $\mathsf{MMS}_n(v_2) = 1/n$. For the latter, intuitively, the total disutility of M can also be evenly distributed by letting each bundle contain $\lfloor \frac{1}{n\alpha} \rfloor = k$ objects with disutility α and one object with disutility $\frac{1}{n} - k\alpha < \alpha$. In total, $kn + n \leq m$ objects are needed. Hence, the disutility function v_3 contains kn objects with disutility α, n objects with disutility $\frac{1}{n} - k\alpha$ and $m - kn - n$ objects with disutility 0, and $\mathsf{MMS}_n(v_3) = 1/n$.

However, when $1/n$ is indivisible by α but the number of objects m is limited to $kn+n-1$, $1/n$ cannot be achieved since some bundles in any n-partition contain no more than k objects, and the disutilities of these bundles are at most $k\alpha < 1/n$. For this case, we show that $\mathsf{MMS}_n(v) \geq k\alpha + \frac{1-kn\alpha}{m-kn}$ for any $v \in \mathcal{V}(\alpha; m)$. Let x be the number of bundles in the n-partition that contain no more than k objects, it follows that $x \geq kn + n - m$. Since the disutility of each of these bundles is at most $k\alpha$, the average disutility of the other bundles is at least

$$\frac{1 - k\alpha x}{n - x} \geq \frac{1 - (kn + n - m) \cdot k\alpha}{m - kn} = k\alpha + \frac{1 - kn\alpha}{m - kn} > k\alpha$$

where the leftmost-hand side is an increasing function of x since $k\alpha < 1/n$, and the last inequality is because $m \geq \lceil \frac{1}{\alpha} \rceil > kn$. Therefore, the largest disutility of any n-partition is at least $k\alpha + \frac{1-kn\alpha}{m-kn}$; that is, $\mathsf{MMS}_n(v) \geq k\alpha + \frac{1-kn\alpha}{m-kn}$ for any $v \in \mathcal{V}(\alpha; m)$. Let v_4 contain kn objects with disutility α and $m - kn$ objects with disutility $\frac{1-kn\alpha}{m-kn} < \alpha$. Clearly, the worst bundle of the best n-partition contains k objects with disutility α and one object with disutility $\frac{1-kn\alpha}{m-kn}$, thus $\mathsf{MMS}_n(v_4) = k\alpha + \frac{1-kn\alpha}{m-kn}$.

A.2 Proof of Lemma 2

That $\Delta_n^{\oplus}(\alpha; m)$ decreases in n is clear by comparing the MinMaxShares of an arbitrary n-partition and the $(n+1)$-partition obtained by adding one empty share. The monotonicity in m (i.e., $\Delta_n^{\oplus}(\alpha; m) \leq \Delta_n^{\oplus}(\alpha; m+1)$) follows that every disutility in $\mathcal{V}(\alpha; m)$ can be transformed to one in $\mathcal{V}(\alpha; m+1)$ by adding an object with disutility 0, without changing the MinMaxShare.

We then show when $m \geq \lceil \frac{2}{\alpha} \rceil - 1$, $\Delta_n^{\oplus}(\alpha; m) \geq \Delta_n^{\oplus}(\alpha; m+1)$, thus $\Delta_n^{\oplus}(\alpha; m)$ remains constant. To achieve this, we first claim that when $m \geq \lceil \frac{2}{\alpha} \rceil - 1$, for any $v \in \mathcal{V}(\alpha; m+1)$ and any allocation $(A_1, \ldots, A_n)$, there exists one bundle such that the total disutility of two of its objects is no more than α. Otherwise, for any bundle A_k, the total disutility of any two objects is larger than α, which means that $v(A_k) > \frac{|A_k|}{2}\alpha$. Upon summing up the lower bounds over all bundles, $1 = \sum_{k \in N} v(A_k) > \frac{\alpha}{2} \cdot \lceil \frac{2}{\alpha} \rceil \geq 1$, a contradiction.

Now we pick any disutility $v \in \mathcal{V}(\alpha; m+1)$, and let $(A_1, \ldots, A_n)$ be the allocation that gives the MinMaxShare of v. By the claim, there exists a bundle (w.l.o.g., A_1) such that two objects $e_1, e_2 \in A_1$ satisfy $v(e_1)+v(e_2) \le \alpha$. We derive a disutility $v' \in \mathcal{V}(\alpha; m)$ by merging e_1 and e_2 into one object e, and show that $\mathsf{MMS}_n(v) = \mathsf{MMS}_n(v')$. On one hand, let $A_1' = A_1 \setminus \{e_1, e_2\} \cup \{e\}$, since $(A_1', \ldots, A_n)$ is an allocation regarding v' with the largest disutility being $\mathsf{MMS}_n(v)$, it follows that $\mathsf{MMS}_n(v) \ge \mathsf{MMS}_n(v')$. On the other hand, by decomposing e into e_1 and e_2, we can convert any allocation regarding v' to an allocation regarding v without changing the largest disutility, thus $\mathsf{MMS}_n(v) \le \mathsf{MMS}_n(v')$.

Therefore, when $m \ge \lceil \frac{2}{\alpha} \rceil - 1$, every disutility in $\mathcal{V}(\alpha; m+1)$ can be transformed to one in $\mathcal{V}(\alpha; m)$ without changing the MinMaxShare, which gives $\Delta_n^{\oplus}(\alpha; m) \ge \Delta_n^{\oplus}(\alpha; m+1)$. By combining the monotonicity in m, $\Delta_n^{\oplus}(\alpha; m)$ remains constant when $m \ge \lceil \frac{2}{\alpha} \rceil - 1$.

B Missing Proofs in Section 3

B.1 Case 3: $n = 2$ and $k = 1$ for unrestricted m

We now prove Corollary 1 for the case of $n = 2$ and $k = 1$, i.e., $\alpha \in D(2,1) \cup I(2,1)$.

Subcase 3.1: $\alpha \in (\frac{1}{5}, \frac{7}{27}]$

When $\alpha \in (\frac{1}{5}, \frac{7}{27}]$, $v(A_1) > \Delta_2^{\oplus}(\alpha) = \frac{3-3\alpha}{4}$. If $E_\alpha \cap A_2 \ne \emptyset$, A_2 contains some objects with disutility α. Notice that Claim 3 holds as long as $k = 1$, thus there exists $S \subseteq A_1$ such that $\frac{1}{3}v(A_1) \le v(S) < v(A_1) - \alpha$. Denote by e^* one object in A_2 with disutility α, Claim 2 gives $v(A_2 \setminus \{e^*\}) \ge v(S) \ge \frac{1}{3}v(A_1)$. As a result, we have

$$1 = v(A_1) + v(A_2) \ge v(A_1) + \frac{1}{3}v(A_1) + \alpha,$$

which gives $v(A_1) \le \frac{3-3\alpha}{4}$, thus contradicting the assumption that $v(A_1) > \Delta_n^{\oplus}(\alpha)$.

Therefore, $E_\alpha \cap A_2 = \emptyset$, which means that all the objects with disutility α are in A_1 and for any $e \in A_2$, $v(e) < \alpha$. We first derive an upper bound and a lower bound of the maximum disutility of the objects in A_2. Denote by e^* one object in A_1 with $v(e^*) = \alpha < v(A_1)$, since $v(A_1) - v(A_2) = 2v(A_1) - 1 > \frac{1-3\alpha}{2}$, Claim 1 gives

$$\max_{e \in A_2} v(e) \le v(e^*) - (v(A_1) - v(A_2)) < \frac{5\alpha - 1}{2}.$$

Notice that $\frac{1-3\alpha}{2} > \frac{\alpha}{3}$ since $\alpha \le \frac{7}{27} < \frac{3}{11}$, $v(A_1) - v(A_2) > \frac{\alpha}{3}$. Then for every $S \subseteq A_2$ with $v(S) < \alpha$, Claim 1 actually gives a tighter bound of $v(S)$, i.e., $v(S) \le v(e^*) - (v(A_1) - v(A_2)) < \frac{2}{3}\alpha$. This also implies that for every $S' \subseteq A_2$ with $v(S') \ge \frac{2}{3}\alpha$, $v(S') \ge \alpha$ actually holds. Let $S^* = \arg\min_{S \subseteq A_2, v(S) \ge \frac{2}{3}\alpha} v(S)$ whose existence is guaranteed since Claim 2 gives $v(A_2) \ge v(e^*) = \alpha$, thus, $v(S^*) \ge \alpha$. Then from the definition of S^*, $v(e) \ge v(S^*) - \frac{2}{3}\alpha \ge \frac{1}{3}\alpha$ holds for any $e \in A_2$, which implies

$$\max_{e \in A_2} v(e) \ge \frac{\alpha}{2}.$$

Otherwise (i.e., $\max_{e \in A_2} v(e) < \frac{\alpha}{2}$), the total disutility of any two objects in A_2 is at least $\frac{2}{3}\alpha$ and smaller than α, which is a contradiction to Claim 1.

We then show that $|A_1|$ is exactly 3. Otherwise (i.e., $|A_1| \ge 4$), there exists $S \subseteq A_1$ such that $v(A_1) > v(S) \ge \alpha + \frac{2}{3}(v(A_1) - \alpha)$. Then Claim 2 gives $v(A_2) \ge v(S) \ge \alpha + \frac{2}{3}(v(A_1) - \alpha)$. Summing up the lower bounds of $v(A_1)$ and $v(A_2)$ leads to a contradiction as below

$$1 = v(A_1) + v(A_2) \ge \frac{5}{3}v(A_1) + \frac{1}{3}\alpha > \frac{15 - 11\alpha}{12} > 1,$$

where the last inequality is because $\alpha \le \frac{7}{27} < \frac{3}{11}$. Therefore, we can denote $A_1 = \{e_1^1, e_2^1, e_3^1\}$ and assume without loss of generality that $v(e_1^1) = \alpha \ge v(e_2^1) = x \ge v(e_3^1) = y$. We then derive the lower bounds of x and y, and reveal the contradiction in this subcase. Since $x \geq y$, the following formula holds

$$x \geq \frac{x+y}{2} = \frac{v(A_1)-\alpha}{2} > \frac{3-7\alpha}{8} \geq \frac{5\alpha-1}{2} > \max_{e\in A_2} v(e),$$

where the second last inequality is because $\alpha \leq \frac{7}{27}$. Then Claim 1 gives the following lower bound of x

$$x \geq \max_{e\in A_2} v(e) + (v(A_1) - v(A_2)) > \frac{\alpha}{2} + \frac{1-3\alpha}{2} = \frac{1-2\alpha}{2}.$$

Claim 1 also gives $y \geq v(A_1) - v(A_2)$. Notice that

$$2 \cdot (v(A_1) - v(A_2)) > \frac{2-6\alpha}{2} > \alpha - \frac{1-3\alpha}{2} > x - (v(A_1) - v(A_2)),$$

where the second inequality is because $\alpha \leq \frac{7}{27} < \frac{3}{11}$, we have the following lower bound of y

$$y > \frac{1}{2} \cdot (x - (v(A_1) - v(A_2))) \geq \frac{1}{2} \cdot \max_{e\in A_2} v(e) \geq \frac{\alpha}{4}.$$

Therefore, $v(A_1) = \alpha + x + y > \alpha + \frac{1-2\alpha}{2} + \frac{\alpha}{4} = \frac{2+\alpha}{4}$, which gives $v(A_1) - v(A_2) = 2v(A_1) - 1 > \frac{\alpha}{2}$. However, according to Claim 1, $v(A_1) - v(A_2) \leq \alpha - \max_{e\in A_2} v(e) \leq \frac{\alpha}{2}$, thus constituting a contradiction.

For the other direction, the disutility function for this subcase contains one object with disutility α and four objects with disutility $\frac{1-\alpha}{4}$. Since $\frac{1}{5} < \alpha \leq \frac{7}{27}$, it follows that $\frac{1-\alpha}{4} < \alpha < 2 \cdot \frac{1-\alpha}{4}$, where the last inequality is because $\alpha \leq \frac{7}{27} < \frac{1}{3}$. Clearly, the MinMaxShare of this disutility function is $3 \cdot \frac{1-\alpha}{4}$.

Subcase 3.2: $\alpha \in (\frac{7}{27}, \frac{2}{7}]$

When $\alpha \in (\frac{7}{27}, \frac{2}{7}]$, $v(A_1) > \Delta_2^{\oplus}(\alpha) = \frac{2+3\alpha}{5}$. If $E_\alpha \cap A_2 \neq \emptyset$, the proof is similar to that for the counterpart of Subcase 3.1. That is, we also have $v(A_1) \leq \frac{3-3\alpha}{4}$, which contradicts $v(A_1) > \Delta_n^{\oplus}(\alpha)$ since $\frac{3-3\alpha}{4} < \frac{2+3\alpha}{5}$ when $\alpha > \frac{7}{27}$.

Therefore, we can focus on $E_\alpha \cap A_2 = \emptyset$. We first derive an upper bound and a lower bound of the maximum disutility of the objects in A_2, which is similar to the counterpart of Subcase 3.1. Denote by e^* one object in A_1 with $v(e^*) = \alpha < v(A_1)$, since $v(A_1) - v(A_2) = 2v(A_1) - 1 > \frac{6\alpha-1}{5}$, Claim 1 gives

$$\max_{e\in A_2} v(e) \leq v(e^*) - (v(A_1) - v(A_2)) < \frac{1-\alpha}{5}.$$

Notice that $\frac{6\alpha-1}{5} > \frac{\alpha}{3}$ since $\alpha > \frac{7}{27} > \frac{3}{13}$, $v(A_1) - v(A_2) > \frac{\alpha}{3}$. Then for every $S \subseteq A_2$ with $v(S) < \alpha$, Claim 1 actually gives a tighter bound of $v(S)$, i.e., $v(S) \leq v(e^*) - v(A_1) - v(A_2) < \frac{2}{3}\alpha$. This also implies that for every $S' \subseteq A_2$ with $v(S') \geq \frac{2}{3}\alpha$, $v(S') \geq \alpha$ actually holds. Let $S^* = \arg\min_{S\subseteq A_2, v(S)\geq \frac{2}{3}\alpha} v(S)$ whose existence is guaranteed since Claim 2 gives $v(A_2) \geq v(e^*) = \alpha$, thus, $v(S^*) \geq \alpha$. Then from the definition of S^*, $v(e) \geq v(S^*) - \frac{2}{3}\alpha \geq \frac{1}{3}\alpha$ holds for any $e \in A_2$, which implies

$$\max_{e\in A_2} v(e) \geq \frac{\alpha}{2}.$$

Otherwise (i.e., $\max_{e\in A_2} v(e) < \frac{\alpha}{2}$), the total disutility of any two objects in A_2 is at least $\frac{2}{3}\alpha$ and smaller than α, which is a contradiction to Claim 1.

Observe that A_1 contains exactly one object with disutility α. Otherwise (i.e., A_1 contains at least two objects with disutility α), Claim 2 gives $v(A_2) \geq 2\alpha$ which leads to the following contradiction

$$1 = v(A_1) + v(A_2) > \frac{2+3\alpha}{5} + 2\alpha > 1,$$

where the last inequality is because $\alpha > \frac{7}{27} > \frac{3}{13}$. Recall that $|A_1| \geq 3$, A_1 contains at least two objects with disutility smaller than α. For each of such objects, we call it a *medium object* if its disutility is larger than $\max_{e\in A_2} v(e)$. Otherwise, we call it a *small object*. Then Claim 1 gives the following lower bound of the disutility of any medium object e

$$\begin{aligned} v(e) &\geq \max_{e\in A_2} v(e) - (v(A_1) - v(A_2)) \\ &= \max_{e\in A_2} v(e) - (2v(A_1) - 1) > \frac{\alpha}{2} - \frac{6\alpha - 1}{5} = \frac{17\alpha - 2}{10}, \end{aligned}$$

as well as the following lower bound of the disutility of any small object e'

$$v(e') \geq v(A_1) - v(A_2) = 2v(A_1) - 1 > \frac{6\alpha - 1}{5}.$$

We then reveal the contradiction by considering possible combinations of objects in A_1 and showing that no possible combination exists.

Combination 1: besides the object with disutility α, A_1 also contains at least 3 small objects. Thus, $v(A_1) > \alpha + 3 \cdot \frac{6\alpha-1}{5} = \frac{23\alpha-3}{5}$. Then a lower bound of the difference between $v(A_1)$ and $v(A_2)$ is

$$v(A_1) - v(A_2) = 2v(A_1) - 1 > \frac{46\alpha - 11}{5} > \frac{\alpha}{2},$$

where the last inequality is because $\alpha > \frac{7}{27} > \frac{22}{87}$. However, according to Claim 1, $v(A_1) - v(A_2) \leq \alpha - \max_{e\in A_2} v(e) \leq \frac{\alpha}{2}$, which is a contradiction. Note that this also implies that except the object with disutility α, the total disutility of the other objects can not exceed that of three small objects. Since the total disutility of one medium object and one small object is larger than

$$\frac{17\alpha - 2}{10} + \frac{6\alpha - 1}{5} = \frac{29\alpha - 4}{10} > \frac{18\alpha - 3}{5} = 3 \cdot \frac{6\alpha - 1}{5},$$

where the inequality is because $\alpha < \frac{2}{7}$, the only combination that remains to consider is that A_1 contains 2 small objects besides the object with disutility α.

Combination 2: besides the object with disutility α, A_1 contains 2 small objects. From the definition of small object, $v(e') \leq \max_{e\in A_2} v(e) < \frac{1-\alpha}{5}$ holds for any small object $e' \in A_1$. Thus, $v(A_1) < \alpha + 2 \cdot \frac{1-\alpha}{5} = \frac{2+3\alpha}{5}$, which is a contradiction to the assumption that $v(A_1) > \Delta_2^{\oplus}(\alpha)$.

For the other direction, the disutility function for this subcase contains one object with disutility α and five objects with disutility $\frac{1-\alpha}{5}$. Since $\frac{1}{6} < \frac{7}{27} < \alpha \leq \frac{2}{7}$, it follows that $\frac{1-\alpha}{5} < \alpha \leq 2 \cdot \frac{1-\alpha}{5}$. Clearly, the MinMaxShare of this disutility function is $\alpha + 2 \cdot \frac{1-\alpha}{5}$.

Subcase 3.3: $\alpha \in (\frac{2}{7}, \frac{1}{3}]$

When $\alpha \in (\frac{2}{7}, \frac{1}{3}]$, $v(A_1) > \Delta_2^{\oplus}(\alpha) = 2\alpha$. If $E_\alpha \cap A_2 \neq \emptyset$, the proof is similar to those for the counterparts of Subcases 3.1 and 3.2. That is, we also have $v(A_1) \leq \frac{3-3\alpha}{4}$, which contradicts $v(A_1) > \Delta_2^{\oplus}(\alpha)$ since $\frac{3-3\alpha}{4} < 2\alpha$ when $\alpha > \frac{2}{7} > \frac{3}{11}$.

Then we focus on $E_\alpha \cap A_2 = \emptyset$. Since $|A_1| \geq 3$, there exists $S \subseteq A_1$ such that $\alpha + \frac{1}{2}(v(A_1) - \alpha) \leq v(S) < v(A_1)$. From Claim 2, we have a lower bound of $v(A_2)$, i.e., $v(A_2) \geq \alpha + \frac{1}{2}(v(A_1) - \alpha)$. Summing up the lower bounds of $v(A_1)$ and $v(A_2)$ leads to a contradiction,

$$1 = v(A_1) + v(A_2) \geq \frac{3}{2}v(A_1) + \frac{\alpha}{2} > \frac{7\alpha}{2} > 1,$$

where the last inequality is because $\alpha > \frac{2}{7}$.

For the other direction, the disutility function for this subcase contains three objects with disutility α and one object with disutility $1 - 3\alpha$ (if $\alpha < \frac{1}{3}$). Since $\alpha > \frac{2}{7} > \frac{1}{4}$, it follows that $1 - 3\alpha < \alpha$. Clearly, the MinMaxShare is 2α.

B.2 Proof of Theorem 1

We now carefully discuss Hill's share when m is not sufficiently large, which completes the proof of Theorem 1. For the sake of contradiction, we assume that there exists a disutility $v \in \mathcal{V}(\alpha; m)$ such that $\mathsf{MMS}_n(v) > \Delta_n^{\oplus}(\alpha; m)$, and let $\mathbf{A} = (A_1, \ldots, A_n)$ be an allocation that gives the MinMaxShare of v. Without loss of generality, assume $v(A_1) \geq \cdots \geq v(A_n)$. We now split the proof into several cases based on the values of n and k, and it suffices to compute the share for the case where m is smaller than the number of objects in the worst-case disutility function in the unrestricted setting.

Case 1: $n \neq 2$ **or** $k \neq 1$

We consider the subcases $\alpha \in D(n,k)$ and $\alpha \in I(n,k)$, separately.

Subcase 1.1: $\alpha \in D(n,k)$

Recall that when $\alpha \in D(n,k)$ with $n \neq 2$ or $k \neq 1$, the disutility function constructed in the setting when m is not restricted contains $kn+n+1$ objects (see Tables 1 and 2). Therefore, if $m \geq kn+n+1$, the tight bound remains unchanged.

Thus we can focus on $m \leq kn+n$. Since $v(A_1) > \Delta_n^{\oplus}(\alpha; m) = (k+1)\alpha$, by Claim 2, $v(A_j) \geq v(A_1) - \alpha > k\alpha$ for any $j \in N \setminus \{1\}$. Moreover, since the disutility of any object is at most α, A_1 contains at least $k+2$ objects and A_j contains at least $k+1$ ones, i.e., $|A_1| \geq k+2$ and $|A_j| \geq k+1$. Accordingly, the total number of objects is at least $k+2+(n-1)(k+1) = kn+n+1 > m$, a contradiction. The disutility function that shows tightness (see Table 4) contains $\lceil \frac{1}{\alpha} \rceil - 1$ objects with disutility α, one object with disutility $1 - (\lceil \frac{1}{\alpha} \rceil - 1)\alpha$, and $m - \lceil \frac{1}{\alpha} \rceil$ objects with disutility 0. This disutility function is valid since $m \geq \lceil \frac{1}{\alpha} \rceil$. Since $\alpha \in D(n,k)$, $\frac{1}{\alpha} \geq \frac{n(k+1)^2+k+2}{k+2} \geq kn+1$, where the last inequality is because $n \geq 0$. Therefore, the disutility function contains at least $kn+1$ objects with disutility α By the pigeonhole principle, the MinMaxShare is at least $(k+1)\alpha$.

Object Disutility	Quantity
α	$\lceil \frac{1}{\alpha} \rceil - 1$
$1 - (\lceil \frac{1}{\alpha} \rceil - 1)\alpha$	1
0	$m - \lceil \frac{1}{\alpha} \rceil$

Table 4: Disutility function for subcase $\alpha \in D(n,k)$ with $n \neq 2$ or $k \neq 1$, and $m \leq kn+n$.

Subcase 1.2: $\alpha \in I(n,k)$

The bound for $\alpha \in I(n,k)$ remains unchanged regardless of the value of m, since there always exists a disutility function whose MinMaxShare is at least $\Delta_n^{\oplus}(\alpha; m) = (k+1)\alpha$. Specifically, the disutility function (see Table 4) also contains $\lceil \frac{1}{\alpha} \rceil - 1$ objects with disutility α, one object with disutility $1 - (\lceil \frac{1}{\alpha} \rceil - 1)\alpha$, and $m - \lceil \frac{1}{\alpha} \rceil$ objects with disutility 0. Since $\alpha \in I(n,k)$, $\frac{1}{\alpha} \geq kn+1$, which means that there are at least $kn+1$ objects with disutility α. By the pigeonhole principle, the MinMaxShare is $(k+1)\alpha$.

Case 2: $n = 2$ **and** $k = 1$

Recall that when $n = 2$ and $k = 1$, $\alpha \in (\frac{1}{5}, \frac{1}{3}]$, thus $m \geq \lceil \frac{1}{\alpha} \rceil \geq 3$. We prove the lemma for this case by considering different values of m and α. When $m = 3$, α can only be $\frac{1}{3}$. The tight bound remains unchanged (i.e., $\Delta_2^{\oplus}(\frac{1}{3}; 3) = \Delta_2^{\oplus}(\frac{1}{3})$), since the disutility function constructed in the unrestricted setting (i.e., Subcase 3.3 in Subsection B.1) contains 3 objects when $\alpha = \frac{1}{3}$.

When $m = 4$, $\alpha \in [\frac{1}{4}, \frac{1}{3})$. Since $v(A_1) > \Delta_2^{\oplus}(\alpha; 4) = 2\alpha$, by Claim 2, $v(A_2) > \alpha$. Therefore, A_1 contains at least 3 objects and A_2 contains at least 2 objects, a contradiction to $m = 4$. For the tightness, the disutility function contains $\lceil \frac{1}{\alpha} \rceil - 1$ objects with disutility α, and one object with disutility $1 - (\lceil \frac{1}{\alpha} \rceil - 1)\alpha$. Sine $\frac{1}{\alpha} > 3$, by the pigeonhole principle, the MinMaxShare is at least 2α.

When $m = 5$, $\alpha \in (\frac{1}{5}, \frac{1}{3}]$. If $\alpha \in (\frac{1}{5}, \frac{7}{27}]$ or $(\frac{2}{7}, \frac{1}{3}]$, the disutility functions constructed in the unrestricted setting (i.e., Subcases 3.1 and 3.3 in Subsection B.1) contain 5 and 4 objects respectively, thus the tight bounds do not change. If $\alpha \in (\frac{7}{27}, \frac{2}{7}]$, since $v(A_1) > \Delta_2^{\oplus}(\alpha; 5) \geq 2\alpha$, by Claim 2, $v(A_2) > \alpha$, thus A_1 contains at least 3 objects and A_2 contains at least 2 objects. More accurately, since $m = 5$, $|A_1|$ is exactly 3 and $|A_2|$ is exactly 2. Moreover, it can be verified that the largest disutility in A_1 is at most the smallest disutility in A_2. Since otherwise, by exchanging one object in A_1 with a strictly larger disutility and one object in A_2 with a strictly smaller disutility, one can get another allocation $\mathbf{A}' = (A_1', A_2')$ such that $v(A_1') < v(A_1)$ and $v(A_2') \leq 2\alpha < v(A_1)$. Let $A_2 = \{e_1, e_2\}$, it follows that $v(e_1) = \alpha$ and $v(e_2) \geq \frac{1}{3} \cdot v(A_1)$. Therefore,

$$v(A_1 \cup A_2) > v(A_1) + \alpha + \frac{1}{3}v(A_1) = \frac{4}{3} \cdot v(A_1) + \alpha.$$

If $\alpha \in (\frac{7}{27}, \frac{3}{11}]$, $v(A_1) > \Delta_2^{\oplus}(\alpha; 5) = \frac{3-3\alpha}{4}$, thus

$$v(A_1 \cup A_2) > \frac{4}{3} \cdot \frac{3 - 3\alpha}{4} + \alpha = 1,$$

a contradiction. If $\alpha \in (\frac{3}{11}, \frac{2}{7}]$, $v(A_1) > \Delta_2^{\oplus}(\alpha; 5) = 2\alpha$, also a contradiction since

$$v(A_1 \cup A_2) > \frac{11}{3}\alpha > 1.$$

The disutility function that shows tightness for $\alpha \in (\frac{7}{27}, \frac{3}{11}]$ is the same as that in Subcase 3.1 in Subsection B.1, i.e., one object with disutility α and four objects with disutility $\frac{1-\alpha}{4}$. Again, since $\frac{1}{5} < \frac{7}{27} < \alpha \leq \frac{3}{11} < \frac{1}{3}$, $\frac{1-\alpha}{4} < \alpha < 2 \cdot \frac{1-\alpha}{4}$, which gives that the MinMaxShare is $\frac{3-3\alpha}{4}$. For $\alpha \in (\frac{3}{11}, \frac{2}{7}]$, the disutility function is the same as that in Subcase 3.3 in Subsection B.1, i.e., three objects with disutility α and one object with disutility $1 - 3\alpha$. Since $\alpha > \frac{3}{11} > \frac{1}{4}$, $1 - 3\alpha < \alpha$, thus the MinMaxShare is 2α.

When $m \geq 6$, $\alpha \in (\frac{1}{5}, \frac{1}{3}]$. Since the disutility functions constructed in the subcases of the unrestricted setting contain no more than 6 objects, thus the tight bounds remain unchanged.

C Missing Materials in Section 5

C.1 Proof of Claim 4

Notice that by Lemma 1 and Lemma 2, $r_n(\alpha; m)$ is weakly increasing in m. Therefore, it suffices to prove the claim for the setting when m is unrestricted, i.e., $r_n(\alpha) \leq \frac{2n}{n+1}$. We first consider the case where $n = 2$ and $k = 1$. In this case, $\alpha \in (\frac{1}{5}, \frac{1}{3}]$. Since $\alpha < \frac{1}{n} = \frac{1}{2}$, $\Delta_2^{\ominus}(\alpha) = \frac{1}{2}$. When $\alpha \in (\frac{1}{5}, \frac{7}{27}]$, $\Delta_2^{\oplus}(\alpha) = \frac{3-3\alpha}{4}$, thus $r_2(\alpha) = \frac{3-3\alpha}{2} < \frac{6}{5} < \frac{4}{3}$; when $\alpha \in (\frac{7}{27}, \frac{2}{7}]$, $\Delta_2^{\oplus}(\alpha) = \frac{2+3\alpha}{5}$, thus $r_2(\alpha) = \frac{4+6\alpha}{5} \leq \frac{8}{7} < \frac{4}{3}$; when $\alpha \in (\frac{2}{7}, \frac{1}{3}]$, $\Delta_2^{\oplus}(\alpha) = 2\alpha$, thus $r_2(\alpha) = 4\alpha \leq \frac{4}{3}$.

We next consider the cases when $n \geq 3$ or $k \neq 1$. When $\alpha > \frac{1}{n}$ which means $\alpha \in I(n, 0)$ or $\alpha \in (\frac{1}{n}, \frac{2}{n+2}] \in D(n, 0)$, $\Delta_n^{\ominus}(\alpha) = \alpha$. Thus, when $\alpha \in I(n, 0)$, $\Delta_n^{\oplus}(\alpha) = \alpha$ and $r_n(\alpha) = 1 < \frac{4}{3} \leq \frac{2n}{n+1}$ since $n \geq 2$; when $\alpha \in (\frac{1}{n}, \frac{2}{n+2}]$, $\Delta_n^{\oplus}(\alpha) = \frac{2\cdot(1-\alpha)}{n}$ and $r_n(\alpha) = \frac{2}{n} \cdot \frac{1-\alpha}{\alpha} < 2 \cdot (1 - \frac{1}{n}) < \frac{2n}{n+1}$. When $\alpha \leq \frac{1}{n}$, it follows that $\alpha \in (\frac{1}{n+1}, \frac{1}{n}] \in D(n, 0)$ or $\alpha \in I(n, k)$ with $k \geq 1$ or $\alpha \in D(n, k)$ with $k \geq 1$. In these cases, $\Delta_n^{\ominus}(\alpha) = \frac{1}{n}$. When $\alpha \in (\frac{1}{n+1}, \frac{1}{n}]$, $\Delta_n^{\oplus}(\alpha) = \frac{2\cdot(1-\alpha)}{n}$ and $r_n(\alpha) = 2 \cdot (1 - \alpha) < \frac{2n}{n+1}$; when $\alpha \in I(n, k) = (\frac{k+2}{n(k+1)^2+k+2}, \frac{1}{kn+1}]$ with $k \geq 1$, $\Delta_n^{\oplus}(\alpha) = (k + 1)\alpha$ and $r_n(\alpha) = n(k + 1) \cdot \alpha \leq \frac{kn+n}{kn+1} \leq \frac{2n}{n+1}$; when $\alpha \in D(n, k) = (\frac{1}{kn+n+1}, \frac{k+2}{n(k+1)^2+k+2}]$ with $k \geq 1$, $\Delta_n^{\oplus}(\alpha) = \frac{k+2}{k+1} \cdot \frac{1-\alpha}{n}$ and $r_n(\alpha) = \frac{k+2}{k+1} \cdot (1 - \alpha) < \frac{kn+2n}{kn+n+1} \leq \frac{3n}{2n+1} < \frac{2n}{n+1}$.

C.2 Proof of Claim 5

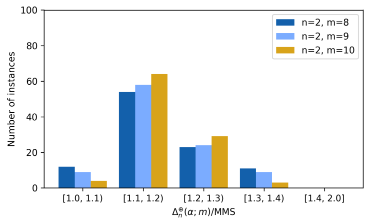
Note that we actually derive the ranges of α that satisfy $r_n(\alpha; +\infty) > \frac{4}{3}$, which are necessary conditions for $r_n(\alpha; m) > \frac{4}{3}$ but may not be sufficient ones. We use the formulas of $r_n(\alpha)$ derived in the proof of Claim 4, and only consider the following cases when $r_n(\alpha)$ may be larger than $\frac{4}{3}$.

- When $\alpha \in (\frac{1}{n}, \frac{2}{n+2}]$, $r_n(\alpha) = \frac{2}{n} \cdot \frac{1-\alpha}{\alpha}$, which is larger than $\frac{4}{3}$ when $\alpha < \frac{3}{2n+3}$. Since $\frac{3}{2n+3} > \frac{1}{n}$ only when $n \geq 4$, the range is $\alpha \in (\frac{1}{n}, \frac{3}{2n+3})$ with $n \geq 4$.
- When $\alpha \in (\frac{1}{n+1}, \frac{1}{n}]$, $r_n(\alpha) = 2 \cdot (1-\alpha)$, which is larger than $\frac{4}{3}$ when $\alpha < \frac{1}{3}$. Since $\frac{1}{n+1} < \frac{1}{3}$ only when $n \geq 3$ and $\frac{1}{n} \leq \frac{1}{3}$ when $n \geq \frac{1}{3}$, the range is $\alpha \in (\frac{1}{n+1}, \frac{1}{n})$ with $n \geq 3$.
- When $\alpha \in I(n,k) = (\frac{k+2}{n(k+1)^2+k+2}, \frac{1}{kn+1}]$ with $k \geq 1$, $r_n(\alpha) = n(k+1) \cdot \alpha$, which is larger than $\frac{4}{3}$ when $\alpha > \frac{4}{3n(k+1)}$. Note that $\frac{4}{3n(k+1)} < \frac{1}{kn+1}$ is equivalent to $(3-k)n > 4$, which can be satisfied only when $k = 1$ or $k = 2$. When $k = 1$, $(3-k)n > 4$ gives $n \geq 3$, $\alpha > \frac{4}{3n(k+1)}$ is equivalent to $\alpha > \frac{2}{3n}$, and $\frac{k+2}{n(k+2)^2+k+2} = \frac{3}{4n+3}$. Since $\frac{3}{4n+3} \geq \frac{2}{3n}$ when $n \geq 6$, the ranges are $\alpha \in (\frac{2}{3n}, \frac{1}{n+1})$ with $3 \leq n \leq 5$, and $\alpha \in (\frac{3}{4n+3}, \frac{1}{n+1})$ with $n \geq 6$. When $k = 2$, $(3-k)n > 4$ gives $n \geq 5$, $\alpha > \frac{4}{3n(k+1)}$ is equivalent to $\alpha > \frac{4}{9n}$, and $\frac{k+2}{n(k+2)^2+k+2} = \frac{1}{4n+1}$. Since $\frac{4}{9n} > \frac{1}{4n+1}$, the range is $\alpha \in (\frac{4}{9n}, \frac{1}{2n+1})$ with $n \geq 5$.
- When $\alpha \in D(n,k) = (\frac{1}{kn+n+1}, \frac{k+2}{n(k+1)^2+k+2}]$ with $k \geq 1$, $r_n(\alpha) = \frac{k+2}{k+1} \cdot (1-\alpha)$, which is larger than $\frac{4}{3}$ when $\alpha < \frac{2-k}{3k+6}$. Note that $\frac{2-k}{3k+6} > 0$ only when $k = 1$. Then, $\alpha \leq \frac{2-k}{3k+6}$ is equivalent to $\alpha < \frac{1}{9}$, $\frac{1}{kn+n+1} = \frac{1}{2n+1}$ and $\frac{k+2}{n(k+2)^2+k+2} = \frac{3}{4n+3}$. Since $\frac{3}{4n+3} \leq \frac{1}{9}$ when $n \geq 6$ and $\frac{1}{9} > \frac{1}{2n+1}$ when $n \geq 5$, the ranges are $(\frac{1}{2n+1}, \frac{3}{4n+3})$ with $n \geq 6$, and $(\frac{1}{2n+1}, \frac{1}{9})$ with $n = 5$.

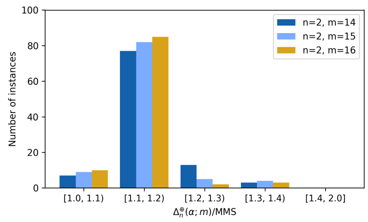
By summarising the above ranges, we complete the proof.

C.3 More Experiments

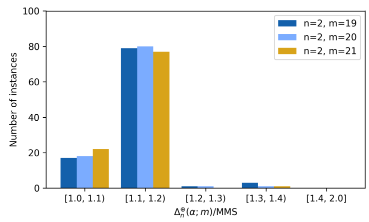
We observe that in Fig. 4, when $n = 2$, the majority of random instances fall into the interval of $[1.1, 1.2)$, in contrast to the other values of n that are concentrated within $[1.0, 1.1)$. This is in part because the ratio of m over n is larger than $n > 2$, given each m. One may be curious that when m becomes larger and larger to n, the majority may be close to the worst-case ratio. Due to this curiosity, we further conduct the following experiment by setting $m = 15 \pm 1$ and $m = 20 \pm 1$, where n is fixed at 2. The results are shown in Fig. 6. As we can see, the instances get more concentrated within $[1.1, 1.2)$, and the number of instances whose ratios are above 1.2 get less and less.



(a) $n = 2$, $m = 8, 9, 10$



(b) $n = 2$, $m = 14, 15, 16$



(c) $n = 2$, $m = 19, 20, 21$

Figure 6: Fixing $n = 2$ and increasing the value of m.