Optimal firm behavior under environmental constraints[1]

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Abstract

The paper examines the Porter and induced-innovation hypotheses in a firm model where: (i) the firm has a vintage capital technology with two complementary factors, energy and capital ; (ii) scrapping is endogenous; (iii) technological progress is energy-saving and endogenous through purposive R&D investment; (iv) the innovation rate increases with R&D investment and decreases with complexity; (v) the firm is subject to emission quotas which put an upper bound on its energy consumption at any date; (vi) energy and capital prices are exogenous. Balanced growth paths are first characterized, and a comparative static analysis is performed to study a kind of long-term Porter and induced-innovation hypotheses. In particular, it is shown that tighter emission quotas do not prevent firms to grow in the long-run, thanks to endogenous innovation, but they have an inverse effect on the growth rate of profits. Some short-term dynamics are also produced, particularly, to analyze the role of initial conditions and energy prices in optimal firm behavior subject to environmental regulation. Among numerous results, we show that (i) firms which are historically "small" polluters find it optimal to massively pollute in the short run: during the transition, new and clean machines will co-exist with old and dirty machines in the productive sectors, implying an unambiguously dirty transition; (ii) higher energy prices induce a shorter lifetime for capital goods but they depress investment in both new capital and R&D, featuring a kind of reverse Hicksian mechanism.

Keywords: Vintage capital, R&D, Emission quotas, Porter hypothesis, Induced-innovation hypothesis, Optimization

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1. Introduction

The arguments for environmental regulation are usually based on what has come to be known as the *Porter hypothesis*. Porter (1991) and Porter and van der Linde (1995) argued that at least in some sectors, a carefully designed environmental regulation as a key feature of industrial policy can increase firm competitiveness by encouraging innovation in environmental technologies. So far, this hypothesis has been the target of numerous studies in several disciplines, including economics, with highly diverging conclusions. In particular, many case studies have been performed, reaching different conclusions depending on firms, industries, countries. An excellent compilation of such case studies can be found in Parto and Herbert-Copley (2007).

A similar hypothesis, popularized by Hicks (1932) and widely applied to environmental economics, especially in its energy part (see Newell, Jaffee and Stavins, 1999, for a seminal contribution), is the so-called *induced-innovation hypothesis*. According to Hicks, the change of relative prices of production inputs stimulates innovation, an innovation of a particular type, directed to save the production factor that becomes relatively expensive. In the context of the energy consumption debate, this hypothesis simply stipulates that in periods of rapidly rising energy prices (relative to other inputs), economic agents will find it more profitable to develop alternative technologies, that is, energy-saving technologies. Just like the Porter hypothesis, the induced-innovation hypothesis in its energy-saving version has been intensively studied in recent years, with again highly diverging outcomes, depending mainly on the aggregation levels considered in the studies. In their well-known work, Newell, Jaffee and Stavins (1999) concluded that a large portion of efficiency improvements in US manufacturing seems to be autonomous, and therefore not driven by the Hicksian mechanism outlined above.

Be it stimulated by tightening environmental regulation, caused by the gradual exhaustion of fossil resources, dictated by international agreements like the Kyoto Protocol or by rapidly increasing energy-prices, the role of innovation at the firm level is the key in the two hypotheses described above. It explains why these hypotheses are actually shaping a substantial part of the environmental literature in economics. If the firms do effectively respond to the latter constraints and circumstances by doing more R&D, then the "environmental problem", understood as the burden involved by environmental constraints on economic development, can be partially solved. This refers to the so-called "Win-Win" outcome mentioned by Porter: innovative firms would not suffer any productivity slump while contributing decisively to a clean environmental and sustainable development.

This paper is devoted to understanding how and under which conditions, if any, firms would engage in R&D investments under environmental constraints and/or rising energy prices. In contrast to numerous papers written in this area (notably in the macroeconomic literature), which typical consider the R&D conducted outside the firms by specialized entities (see, for example, Hart, 2004), we start with the key assumption that firms, confronted with environmental constraints, may decide to individually engage in R&D activities. We do consider such an extension as essential to get through the puzzle, and there are several reasons for this approach to be preferred:

i) First of all, the role of "production" firms in the development of clean technologies cannot be under-scored because most environmental problems are firm or industry specific and cannot be simply solved by importing technologies. We shall develop this idea in the next section when describing the concrete case of the chlor-alkali industry in Japan (Yarime, 2007).

ii) Second, it has been repeatedly established that at least in the case of large corporations (see Carraro and Siniscalco, 1994), firms tend to respond to environmental policy measures through innovations, not by switching inputs or reducing output.

iii) Last but not least, as mentioned by several authors (among them, Carraro and Siniscalco, quoted just above), very high taxes are needed to bring down CO_2 emissions in the absence of innovations. This justifies the approach taken in this paper: understanding how the firms (for example, subject to pollution quotas) engage individually in R&D is indeed a key task.

Throughout this paper, we shall consider vintage capital technologies. Capital goods produced at different dates embody different technologies, the youngest vintages are the most energy-saving, and, therefore, the least polluting. Beside realism, working with vintage capital production functions allows to capture some key elements of the problem under consideration, which would be lost under the typical assumption of homogenous capital. For instance, facing an emission tax, firms are tempted to downsize. However, in a typical framework where the firm also chooses the optimal age structure of capital, which is the main additional control variable in vintage capital models, downsizing entails modernization: the older and, thus, the dirtier machines and technologies are then removed. For productivity analysts, this is good news: contrary to the typical framework with homogenous capital, we have a clear productivity-enhancing effect of emission taxes in such a framework, thus giving a chance to the Porter ``Win-Win´´ outcome to arise, even in the absence of firms' innovative activities.

Indeed, whether such an indirect modernization effect can compensate the so-called profit-emission effect according to which profits decline under emission taxes sounds as a highly intriguing question. Very few papers have tried to deal with this issue so far, manly due to the sophisticated mathematical structure of vintage capital models. Two valuable exceptions should be mentioned here. Xepapadeas and de Zeeuw (1999) provided the first inspection into this problem. They concluded that the costs of environmental regulation were mitigated if firms responded to emission taxes by scrapping the older and dirtier technologies. Therefore, the indirect modernization effect offsets a substantial part of the negative profit-emission negative effect, but not totally. Feichtinger, Hartl, Kort and Veliov (2005) introduced a better specification of embodied technological progress underlying the considered vintage capital structure. They concluded that if learning costs are incorporated into the analysis (that's running new machines at their full productivity potential takes time), then the magnitude of the modernization effect is strongly reduced, and environmental regulation has a markedly negative effect on industry profits.

Our paper extends the two previous papers, where the pace of technological progress is kept exogenous, and endogenizes R&D decisions. We have already justified largely why this endogenization is necessary for a proper appraisal of the ``environmental problem" as defined above. We shall refine our arguments in this respect in the next factual section.

We characterize optimal firm behavior both asymptotically and in the long-run, and we extract several new results, thanks to the endogenous nature of technological progress. In particular, we outline here three crucial results:

i) In the long-run, tighter emission quotas coupled with liquidity constraints do not prevent firms from growing in the long-run, thanks to endogenous innovation, and this is good news. However, these constraints have an inverse effect on the growth rate of profits. In other terms, while R&D is crucial for firms to keep on growing despite environmental and financial constraints, we get the natural outcome (at least, at the firm level) that no Porter-hypothesis is expected to arise in the long-run, namely, strengthening environmental regulation does not improve the situation of the firms in the long-run, under the conditions of the model (price-taking liquidity-constrained firms).

ii) In the short-run, the results are even clearer. For example, we establish that firms which are historically "small" polluters find it optimal to massively pollute in the short run: during the transition, new and clean machines will co-exist with old and dirty machines in the productive sectors, implying an unambiguously dirty transition. Therefore, the model provides micro-foundations for an essential part of the so-called Environmental Kuznets Curve.

iii) Last but not least, we show that under some specific but reasonable circumstances, higher energy prices induce shorter lifetime for capital goods but they depress investment in both new capital and R&D, featuring a kind of reverse Hicksian mechanism.

The paper is organized as follows. Section 2 is devoted to describing some salient characteristics of the ”environmental regulation” taken at the concrete firm level, borrowing from the writing of some technologists. Section 3 formally describes our firm optimization problem and outlines some of its peculiarities. Section 4 derives the optimality conditions and interprets them. Section 5 is concerned with the long-term optimal behavior of firms and Section 6 presents some implications for optimal short-term dynamics. Section 7 concludes.

2. Insight from technologists

We start with a short description of the case of the chlor-alkali Japanese industry, which is in our view an excellent illustration of firm’s behavior under environmental regulation in an energy-saving context. We then switch to other salient features of the problem, as depicted by several technologists.

2.1. An illustration: the chlor-alkali industry in Japan

This sub-section is entirely based on Yarime (2007). The chlor-alkali industry produces chlorine and caustic soda through electrolysis. Because it involves electrolysis, it is one of the major energy consumers in the Japanese industry.[5] In this context, a major concern

[5] Yarime (2007) reports that about 3% of total industry electricity consumption in Japan can be attributed to the chlor-alkali industry in 1996, which also accounts for about one-fifth of total chemical industry in this year.

of the firms operating in this industry is to develop innovative techniques in order to reduce energy consumption. Of course, the R&D activities conducted to this end were not all dictated by environmental constraints or rising energy prices. This was certainly not the case in the 60s for example. On the other hand, the technological context of such an industry is highly interesting for the study of energy-saving innovation processes.

To this context, one has to add a sensitive environmental issue, linked to the electrolysis technique used, which has motivated an increasingly severe environmental regulation from the late 60s. Indeed, at that time, the electrolytic process employed was a mercury process, thus based on a highly toxic substance. It was relatively quickly established that the mercury released by the chlor-alkali industry to the neighboring seas was the cause of the so-called Minimata disease, which caused about 700 victims in that time.[6] The Japanese authorities started ruling against chlor-alkali industry from the mid-60s, stipulating among others quantitative limits to control the levels of mercury released to environment. In 1974, the Japanese authorities took a step further against the industry and require the conversion of as many mercury plants as possible to the unique alternative at that time, the made-in-USA diaphragm electrolytic process, by the end of 1975.[7]

Now, comes the most interesting part of the story. Because the alternative diaphragm process was clearly disadvantageous in terms of energy consumption compared to the mercury process, and given the period of rapidly increasing energy prices (recall the anti-mercury process regulation was taken during the first oil crisis), it quickly appeared to

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[6] Minimata disease refers to Minimata Bay in the Southern part of Japan, where the first cases of mercury poisoning were discovered.

[7] Interestingly, as mentioned by Yarime (2007), the final decision to rule out the mercury process was taken when the process was accounting for 95% of total capacity, which of course created heavy tensions between the producers and the Japanese authorities. See more in Yarime's contribution.

both producers and authorities that there was an urgent need to develop an alternative electrolytic process, less energy-consuming than the diaphragm process and less polluting than the mercury process.[8] This motivated a massive R&D effort in developing a third electrolytic process, the ion exchange membrane process, and the suspension by May 1977 of the conversion program (to the diaphragm process technology). As mentioned by Yarime (2007), although the idea of using ion exchange membranes had been known by many years at that time, a significant R&D effort was needed to develop ion exchange membranes adapted to the chlor-alkali industry, and the number of patent applications by Japanese firms increased markedly after the mid-70s and until the early 80s in this field. In 1998, about 90% of the Japanese chlor-alkali plants used the ion exchange membrane process.

2.2. Other features

Several other insights can be gained from the technology literature concerning the innovative processes in the industry subject to environmental constraints. We shall mention two of them, which will be explicitly considered in our theoretical set-up.

i) **The role of financial constraints:** This type of constraints is, of course, crucial as long as one is concerned with technological renovation, especially when it is imposed by law. If the firms do not face any type of financial constraints, then they could finance R&D expenditures with no limit, which is certainly unrealistic. In the case of the Japanese chlor-alkali industry described above, financial constraints are even more crucial since the whole industry was required to switch

[8] Yarime also mentioned some problems related to the poor quality of the caustic soda produced by the diaphragm technique. This point goes beyond our framework but it is certainly highly intriguing.

technology in a limited amount of time (see the very interesting description of the debate between the chlor-alkali industry and the Japanese authorities on the financing of the required R&D programs in Yarime's paper).

ii) **The role of technological complexity:** It is very well known that the success of R&D programs depends, among others, on the complexity and sophistication of the technologies to be up-graded. Complexity is therefore a fundamental ingredient of early technology adoption theories *à la* Nelson and Phelps (1964) and of more recent standard growth theory (see for example, Barro and Sala-i-Martin, 1995, chapter 7, or Segerstrom, 2000). Needless to say, the problem of technological sophistication is also a sensitive barrier to technological progress because of limited amount of available skills and hi-tech capital (see Chudnowsky and Lopez, 2007, pp 88-121, for the Argentinian case).

We shall take these aspects into account in the firm generic problem addressed hereafter.

3. The firm problem

We shall consider the problem of a firm seeking to maximize the net profit that takes into account the energy consumption $E(t)$, the investment $R(t)$ to R&D, and the investment $\mu(t)$ into new capital:

$$I = \int_0^\infty e^{-rt}[Q(t) - p(t)E(t) - R(t) - k(t)\mu(t)]dt \longrightarrow \max_{\mu,a,R} \qquad (1)$$

where $k(t)$ is the given unit capital price (per capacity unit), $p(t)$ is the given energy price, e^{-rt} is the discounting factor. Here $Q(t)$ is the total product output at t,

$$Q(t) = \int_{a(t)}^{t} \mu(\tau)d\tau, \qquad (2)$$

$$c(t) = Q(t) - p(t)E(t) - R(t) - k(t)\mu(t) \quad (3)$$

is the net profit or cash flow. We therefore postulate a Leontief vintage capital production function as in Boucekkine, Germain and Licandro (1997, 1999) or Hritonenko and Yatsenko (1996, 2005). In equation (2), $a(t)$ measures the vintage index of the oldest machine still in use at time t, or in other words, t-$a(t)$ measures the scrapping time at date t. The whole complexity of the optimization problem considered in this paper comes from the fact that a is a control variable, which is quite unusual in economic theory. We shall come back to this point in detail later. For now, let us notice that we do not assume any output-augmenting (embodied or disembodied) technological progress: whatever the vintage τ is, all machines produce one unit of output. In our framework, the technological progress is exclusively energy-saving, which is the key component of the debate around technological progress and environmental sustainability.

In contrast to the related literature (notably to Feichtinger et al., 2005, 2006 and 2007), we assume that firms invest in R&D. It reflects the fact that the environmental problems (here linked to energy consumption and subsequent CO_2 emissions) are firm-specific, so, the firms cannot simply import preexisting cleaner technologies. And even if a relevant technology could be imported (like the diaphragm technique in our Japanese industry case), a costly adoption work is needed. Let us call $\beta(\tau)$ the level of the energy-saving technological progress at date t. We postulate that this level evolves endogenously according to:

$$\frac{\beta'(\tau)}{\beta(\tau)} = \frac{f(R(\tau))}{\beta^d(\tau)}, \qquad d > 0, \quad (4)$$

where f is increasing and concave: $df/dR>0$, $d^2f/dR^2<0$. Equation (4) deserves a few comments. It basically stipulates that the rate of energy-saving technical progress is an increasing (and concave) function of the R&D effort and a decreasing function of its level. The latter specification is designed to reflect the negative impact of technological complexity on R&D success. The parameter *d* measures the extent to which complexity impacts the rate of technological progress (just like in Segerstrom, 2000, for example). It will play an important role hereafter, which is consistent with the available evidence, mentioned in Section 2, on the role of technological complexity in the adoption of (clean) technologies.

We also assume that the energy-saving technological progress is fully embodied in new capital goods, which implies, keeping the Leontief structure outlined above, that total energy consumption is given by

$$E(t) = \int_{a(t)}^{t} \frac{\mu(\tau)}{\beta(\tau)} d\tau. \tag{5}$$

We now introduce the environmental constraint to the firm, through a simple emission quota constraint:

$$E(t) \leq E_{\max}(t), \tag{6}$$

where the regulation function $E_{\max}(t)$ is given. Implicit in our setting, the consumption of of energy is the sole source of pollution through CO_2 emissions. Restricting energy consumption is therefore a direct way to limit pollution. The firms are also subject to a second type of constraint, financial constraint, which we also model in a straightforward way by imposing the positivity of cash-flows, $c(t)$, at any date t, as we will see later.

Let us now summarize the optimal control problem to tackle.

The *unknown functions* are:

- the investment $\mu(t)$, $\mu(t) \geq 0$, into new capital (measured in the **capacity units**)

- the R&D investment $R(t)$, $R(t) \geq 0$, and the technology $\beta(t)$,
- the capital scrapping time $t-a(t)$, $a'(t) \geq 0$, $a(t) < t$,
- the output $Q(t)$, cash-flow $c(t)$, and energy consumption $E(t)$, $t \in [0,\infty)$.

The *constraints* are given by the environmental constraint (6), plus the positivity, liquidity constraint and other regularity conditions:

$$R(t) \geq 0, \quad c(t) \geq 0, \quad \mu(t) \geq 0, \quad a'(t) \geq 0, \quad a(t) < t, \tag{7}$$

The condition $a'(t) \geq 0$ is a standard constraint in vintage capital models implying that scrapped machines cannot be reused. We shall also specify the *initial conditions* as follows:

$$a(0) = a_0 < 0, \quad \beta(a_0) = \beta_0, \quad \mu(\tau) \equiv \mu_0(\tau), \quad R(\tau) \equiv R_0(\tau), \quad \tau \in [a_0, 0]. \tag{8}$$

The optimal control problem (1)-(8) has several mathematical peculiarities (compared to the typical optimal control problem in economics), which makes it quite hard to tackle. We come back to the technical part in the next Section 4 where the necessary optimality conditions are developed. Before, let us stress the following economic aspects:

i) **Technological progress modelling 1:** Our formalization simplifies to a manageable mathematical complexity the chlor-alkali industry example described in Section 2. In particular, while the incentives to develop alternative technologies were driven by distinct purely ecological motivations (get rid of the mercury-based electrolytic process) and energy-saving reasons, the two motivations are merged in our modelling: the firm aims at developing energy-saving technologies to lower its energy expenditures and to cope with environmental regulation. On the other hand, while the technological menus seem to be limited to three in the chlor-industry case (mercury, diaphragm and ion exchange membrane processes), the R&D effort was actually continuous in time, resulting in a non-lumpy trajectory of patents as documented in Yarime (2007). Hence, our continuous time setting is still adequate.

ii) **Technological progress modelling 2:** In our modelling, technological improvements affect only the new capital goods. This is crystal clear in equation (5) giving total energy consumption. Of course, this need not be the case in general. A part of energy-saving innovations is probably disembodied, and a more general formulation of the problem taking into account this aspect would, in particular, replace the ODE on $\beta(t)$, by a PDE on $\beta(\tau,t)$. This extension is out of the scope of this paper. As one can guess, our optimal control problem (1)-(8) is already extremely tricky. Moving to PDEs specifications of technological progress would oblige us to resort massively to numerical simulation (as in Feichtinger et al., 2006), which we want precisely to avoid. On the other hand, part of technological innovations in the workplace are of course imported, but nevertheless hardly at zero cost. Therefore, they cand be ``imported'' in the technological variable R(t) without any decisive loss of generality.

iii) **Technological progress modelling 3:** As in Hart (2004), we can extend the model by distinguishing between R&D devoted to increase output, and R&D environmental-friendly. This might probably change some of the results of the paper. Given the induced algebraic cost, we have decided to restrict our attention to energy-saving technological progress, which also happens to be the environmental-friendly innovations in our set-up.

iv) **Environmental regulation:** In this paper, we focus on emission quotas, as this seems to be one of the salient characteristics of environmental regulation both at the national and transnational levels. Other policy instruments could have been considered like emission taxes (see Xepapadeas and De Zeeuw, 1999) for example. Moreover, it could be interesting to compare policy tools regarding the fulfillment of the Porter hypothesis. This goes beyond the scope of this paper.

4. Extremum conditions

We now move the derivation of the optimality conditions. For mathematical convenience, we change the unknown (decision) variable $\mu(t)$ to

$$m(t) = \mu(t)/\beta(t), \tag{9}$$

which is also the investment into new capital (but measured in the *energy consumption units* rather than in *capacity units*). In the variables R and m, the optimization problem (1)-(8) becomes

$$I = \int_0^\infty e^{-rt}[Q(t) - p(t)E(t) - R(t) - k(t)\beta(t)m(t)]dt \longrightarrow \max_{R,m,a} \tag{10}$$

$$c(t) = Q(t) - p(t)E(t) - R(t) - k(t)\beta(t)m(t), \tag{11}$$

$$Q(t) = \int_{a(t)}^{t} \beta(\tau)m(\tau)d\tau, \tag{12}$$

$$E(t) = \int_{a(t)}^{t} m(\tau)d\tau, \qquad E(t) \le E_{\max}(t), \tag{13}$$

$$R(t)\ge 0, \quad m(t)\ge 0, \quad c(t) \ge 0, \quad a'(t) \ge 0, \quad a(t) < t, \tag{14}$$

$$a(0) = a_0<0, \quad \beta(a_0)=\beta_0, \quad m(\tau) \equiv m_0(\tau), \quad R(\tau)\equiv R_0(\tau), \quad \tau\in[a_0, 0]. \tag{15}$$

The substitution (9) removes $\beta(t)$ from equation (5) and adds it to the last term in the functional (10). Equation (4) for the unknown $\beta(t)$ remains the same. In the case $d>0$, the solution of (4) has the form:

$$\beta(\tau)=\left(d\int_0^{\tau} f(R(v))dv+B^d\right)^{1/d}, \tag{16}$$

where the constant $B=\beta(0)=\left(d\int_{a_0}^{0} f(R_0(v))dv+\beta_0{}^d\right)^{1/d}$ is uniquely determined by the initial conditions (8) or (15). From now on, we work with the following explicit specification for endogenous technological progress:

$$f(R)=bR^n, \quad 0<n<1, \quad b>0. \tag{17}$$

By (4), this implies that the elasticity of the rate of technological progress with respect to R&D expenditures is constant and equal to n. The larger is n, the bigger is the efficiency of investing in R&D.

The optimization problem (*OP*) (10)-(17) includes seven unknown functions R, β, m, a, Q, c, and E connected by four equalities (11), (12), (13), and (16). Following Hritonenko and Yatsenko (1996), Yatsenko (2004), and Yatsenko and Hritonenko (2005), we will choose R, m, and a' as the *independent* variables (or controls) of the OP and consider the rest of the unknown functions β, m, a, Q, c, and E as the *dependent* (*state*) variables.

The majority of optimization models of mathematical economics are treated using FOC (first-order conditions) for *interior trajectories only*. In contrast, the nature of the OP (10)-(17) requires taking into account the inequalities $E(t)\le E_{max}(t)$, $R(t)\ge 0$, $m(t)\ge 0$, $a'(t)\ge 0$, $a(t)<t$, and $c(t)\ge 0$ on unknown variables in the constraints (13) and (14). These inequalities have an essential impact on extremum conditions and optimal dynamics and

are treated differently in the below analysis. The inequalities $R\geq 0$ and $m\geq 0$ are the standard constraints on control variables, which are common in the optimization theory. The non-standard constraints $a'(t)\geq 0$ and $a(t)<t$ are handled following the technique developed by Hritonenko and Yatsenko in several papers already cited. The constraint $E\leq E_{\max}$ is considered in two cases of Theorem 1 below. Finally, the constraint $c\geq 0$ is the most inconvenient mathematically and is checked *a posteriori* (see Remark 2 below).

Let the given functions p, k, and $E_{\max}$ be continuously differentiable, and m_0 and R_0 be continuous. To keep the OP statement correct, the smoothness of the unknown variables should be consistent. We will assume that the decision variables R and m (and a' when necessary) are measurable *almost everywhere* (*a.e.*) on $[0,\infty)$. Then, the unknown state variables a, c, Q, and E in (10)-(15) are a.e. continuous on $[0,\infty)$, as established in Hritonenko and Yatsenko (2006). We also assume a priori that the improper integral in (10) converges (it will be true in all subsequent Theorems 2-4).

The *necessary condition for an extremum* (*NCE*) in the OP (10)-(17) is given by the following statement

Theorem 1. *Let $(R^*(t), m^*(t), a^*(t), \beta^*(t), Q^*(t), c^*(t), E^*(t),)$, $t\in[0,\infty)$, be a solution of the OP (10)-(17).*

(A) *If $E^*(t)=E_{max}(t)$ and $c^*(t)>0$ at $t\in\Delta\subset[0,\infty)$, and $E_{max}'(t)\leq 0$, then*

$$I_R'(t)\leq 0 \text{ at } R^*(t)=0, \qquad I_R'(t)=0 \text{ at } R^*(t)>0, \tag{18}$$

$$I_m'(t)\leq 0 \text{ at } m^*(t)=0, \qquad I_m'(t)=0 \text{ at } m^*(t)>0, \quad t\in\Delta, \tag{19}$$

where

$$I_R'(t) = bnR^{n-1}(t)\int_t^{\infty} \beta^{1-d}(\tau)m(\tau)\left[\frac{e^{-r\tau} - e^{-ra^{-1}(\tau)}}{r} - e^{-r\tau}k(\tau)\right]d\tau - e^{-rt}, \qquad (20)$$

$$I_m'(t) = \int_t^{a^{-1}(t)} e^{-r\tau}\left[\beta(t) - \beta(a(\tau))\right]d\tau - e^{-rt}\beta(t)k(t), \qquad (21)$$

the state variable a(t) is determined from (13), $a^{-1}(t)$ is the inverse function of a(t), and

$$\beta(\tau) = \left(db\int_0^{\tau} R^n(\xi)d\xi + B^d\right)^{\frac{1}{d}}. \qquad (22)$$

(B) *If $E^*(t)<E_{max}(t)$ and $c^*(t)>0$ at $t\in\Delta$, then*

$$I_R'(t)\le 0 \text{ at } R^*(t)=0, \qquad I_R'(t)=0 \text{ at } R^*(t)>0,$$

$$I_m'(t)\le 0 \text{ at } m^*(t)=0, \qquad I_m'(t)=0 \text{ at } m^*(t)>0, \qquad (23)$$

$$I_{a'}'(t)\le 0 \text{ at } da^*(t)/dt=0, \qquad I_{a'}'(t)=0 \text{ at } da^*(t)/dt>0, \quad t\in\Delta,$$

where

$$I_m'(t) = \int_t^{a^{-1}(t)} e^{-r\tau}\left[\beta(t) - p(\tau)\right]d\tau - e^{-rt}\beta(t)k(t), \qquad (24)$$

$$I_{a'}'(t) = \int_t^{\infty} e^{-r\tau}[p(\tau) - \beta(a(\tau))]m(a(\tau))d\tau, \qquad (25)$$

$I_R'(t)$ is as in (20), and $\beta(t)$ is as in (22).

The proof is very long and technical and we report all the details in the Appendix. The expressions (20), (21), (24), and (25) are the *Freshet derivatives* of the functional I in variables R, m, and a'. The derivative $I_m'(t)$ has different forms (21) and (24) depending on whether the restriction (13) is active or inactive. Before giving the economic interpretation of the optimality conditions, some technical comments are in order.

Remark 1. If (13) is active (Case A), then the state variable a is determined from $m(a(t))a'(t)=m(t)-E_{\max}'(t)$ and the state restriction $a'\geq 0$ on the variable a in (14) is satisfied if $E_{\max}'(t)\leq 0$, $t\in[0,\infty)$. If the condition $E_{\max}'(t)\leq 0$ fails for some $t\in\Delta\subset[0,\infty)$, then Theorem 1 is still valid in Case A if we replace the differential constraint $a'(t)\geq 0$ in (14) with the stricter constraint $m(t)\geq \max\{0, E_{\max}'(t)\}$ on the control m (see Hritonenko and Yatsenko, 2006, for a proof).

Remark 2. To keep mathematical complexity reasonable, we do not include the constraint $c(t)\geq 0$ into the NCE. To be complete, Theorem 1 has to include two more cases: $E^*>E_{\max}$, $c^*=0$, and $E^*=E_{\max}$, $c^*=0$. The problem (10)-(17) in these cases should be treated as an OP with state constraints, which leads to significant mathematical challenges (see Hartl, Sethi and Vickson, 1995, for an insight into this issue). As we shall see, the regime $c^*(t)=0$ does not usually appear in the long-term dynamics (Section 5) and may have an impact only on the transition dynamics as one of possible scenarios (Section 6).

Remark 3. Sufficient conditions for an extremum for such OPs are complicated and involve the second Freshet derivatives of the functional I. The authors derived and analyzed such condition in the form $J=\begin{vmatrix} I''_{RR}(t) & I''_{Rm}(t) \\ I''_{mR}(t) & I''_{mm}(t) \end{vmatrix}<0$ at $R=R^*$, $m=m^*$ for the Case (A) with active restriction (13). It is not included into this paper.

Remark 4. The vintage models with endogenous TC are multi-extremal under natural conditions, see Chapter 6 in Hritonenko and Yatsenko (1996). We can show that the OP (10)-(17) may also possess two *local* extrema:

(1) *the trivial solution* $R^0(t)\equiv 0$, $m^0(t)\equiv 0$, $a_0\leq a^0(t)\leq 0$, $t\in[0,\infty)$. The solution is verified by its substitution into (20),(24),(25), then $I_R'(t)<0$, $I_m'(t)<0$, and $I_{a'}'(t)<0$, i.e., the NCE (23) holds. This local solution describes economic dynamics with no investment to technological renovation when the entire profit goes to the consumption goods. The trivial solution is not stable in the sense that some (small) positive investments in new capital and R&D can force the economic system to jump to the next solution.

(2) *the non-trivial solution*, where $R^*(t)$, $m^*(t)$, $a^*(t)$ are positive, at least, on some parts of the planning horizon $[0,\infty)$. It describes the case where the economic system installs new equipment and invests into science and technology.

The paper focuses on the structure of the *non-trivial* solution (R^*, m^*, a^*).

Let us move now to some economic interpretations of the obtained first-order optimality conditions. In order to compare more easily with the existing literature, we start with the case (A), that is, when the environmental constraint is binding. Indeed, in such a case, the binding environmental constraint can be broadly viewed as an "equilibrium" condition in the energy market, where the quota plays the role of supply. Let us interpret the optimality conditions with respect to investment and R&D, the case of scrapping being trivially fixed by Remark 1 above. Using equations (19) and (21), the (interior) optimal investment rule may be rewritten as:

$$\int_{t}^{a^{-1}(t)} e^{-r\tau}\left[1-\frac{\beta(a(\tau))}{\beta(t)}\right]d\tau = e^{-rt}k(t)$$

The interpretation of such a rule is quite natural having in mind the early vintage capital literature (notably Solow et al., 1966, and Malcomson, 1975) as exploited in Boucekkine, Germain and Licandro (1997). In our model, one unit of capital at date t costs $k(t)$ or $e^{-rt}k(t)$ in present value. This is the right-hand side of the optimal rule above. The left-hand side should therefore give us the marginal benefit from investing. Effectively, it is the integral of discounted gains from investing over the lifetime of a machine bought at t (since $a^{-1}(t)$ is by construction the lifetime of such a machine). At any date comprised between t and $a^{-1}(t)$, a machine bought at t will provide one unit of output but the firm has to pay the corresponding energy expenditures $\frac{\beta(a(\tau))}{\beta(t)}$. Given our Leontief specifications, $\frac{1}{\beta(t)}$ is the energy requirement of any machine bought at date t. $\beta(a(\tau))$ plays therefore the role of the effective price of the input paid by the firm. How could this be rationalized? Simply by noticing that under a binding environmental constraint, the latter mimics a clearing market condition as in the early vintage macroeconomic literature (see for example, Solow et al., 1966).[9] In such a framework, the marginal productivities of energy should be equalized across vintages, implying a tight connection between the effective price of energy and the energy requirement of the oldest machine still operated. More precisely, the latter price, which happens to be the Lagrange multiplier associated to the binding environmental constraint, is equal to the inverse of the energy requirement of the oldest machine still in use, which is equal to $\beta(a(\tau))$ at any date τ comprised between t and $a^{-1}(t)$. Notice that in such a case, the effective price of energy $\beta(a(\tau))$ is not generally equal to $p(t)$. The latter does not play any role since energy expenditures become predetermined equal to $p(t)E_{\max}(t)$ in the constrained regime. Things are completely different in the case where the environmental constraint is not binding (case B of Theorem 1). In such a case, the optimal investment rule becomes (following equation (24)):

$$\int_{t}^{a^{-1}(t)} e^{-r\tau}\left[1-\frac{p(t)}{\beta(t)}\right]d\tau = e^{-rt}k(t),$$

and $\beta(a(\tau)) = p(t)$ as in the firm problem studied by Malcomson (1975) (with again labor playing the role of energy), making a clear difference with respect to the constraint case A. Our framework thus extends significantly the benchmark theory to allow for situations in which input markets do not necessarily clear due to institutional constraints.

Let us interpret now the R&D optimal rule, which is also new in the literature. Using (20), it is given by

[9] In Solow et al., the role of energy is played by labor.

$$bnR^{n-1}(t)\int_{t}^{\infty} \beta^{1-d}(\tau)m(\tau)\left[\frac{e^{-r\tau}-e^{-ra^{-1}(\tau)}}{r}-e^{-r\tau}k(\tau)\right]d\tau = e^{-rt}$$

The right-hand side is simply the present value of marginal investment in R&D. The marginal benefit is given by the left-hand side. Contrary to the optimal investment rule, the gains from doing R&D last forever: the R&D investment induces a knowledge accumulation process, which is not subjected to obsolescence in our case, in contrast to capital goods. The integrand can be understood if one has in mind the maximized function (10) in the form

$$I = \int_{0}^{\infty} e^{-rt}\left[\int_{a(t)}^{t} \beta(\tau)m(\tau)d\tau - p(t)\int_{a(t)}^{t} m(\tau)d\tau - R(t) - k(t)\beta(t)m(t)\right]dt$$

and the given endogenous law (16),(17) of motion of technological progress. It should be noticed that rewriting the problem in terms of $m(t)$, rather than in terms of investment in physical units $\mu(t)$, does not mean rewriting a problem with input-saving technical progress as a problem with output-augmenting technical progress. As one can see, at fixed $m(t)$, an increase of $R(t)$ (and, therefore, $\beta(t)$) increases not only the output $Q(t)$ but also the investment expenditures through the term $k(t)\beta(t)m(t)$. The left-hand side of the optimal R&D rule takes precisely into account this trade-off. On one hand, the marginal increase in $\beta(\tau)$, $\tau \geq t$, following the marginal rise in $R(t)$, that is $\dfrac{bnR^{n-1}(t)}{\beta^{d}(\tau)}$, impacts positively output by improving the efficiency of all vintages after date t. Notice that since machines have a finite lifetime, this effect should be computed between τ and $a^{-1}(\tau)$ for each vintage τ, which explains the factor $\dfrac{e^{-r\tau}-e^{-ra^{-1}(\tau)}}{r} = \int_{\tau}^{a^{-1}(\tau)} e^{-rs}ds$ in the integrand. On the other hand, the rising $\beta(t)$ increases investment expenditures (for a fixed $m(t)$), which explains the negative term, $e^{-r\tau}k(\tau)$, in the integrand.

We now move to the study of the system of the optimality conditions extracted above. We first start by seeking for exponential solutions (for naturally growing variables like *R(t)*), the so-called balanced growth paths (Section 5), which can feature a kind of long-term dynamics, then we move to short-term dynamics (Section 6).

5. Analysis of optimal long–term dynamics.

The optimal long–term dynamics of the OP can involve interior regimes such that $I_R'\equiv 0$ and $I_m'\equiv 0$. Let us assume that the environmental constraint (13) is active in the long run: $E(t)=E_{\max}(t)$ at $t\in[t_l, \infty)$, $t_l\geq 0$ (we will study the alternative case later). The corresponding long-term interior regime $(R_\Lambda, m_\Lambda, a_\Lambda)$ is determined by the system of three nonlinear integral equations

$$I_R'(t)=0, \qquad I_m'(t)=0,$$

$$\int_{a(t)}^{t} m(\tau)d\tau = E_{\max}(t), \quad t\in[t_l, \infty). \tag{26}$$

where $I_R'(t)$ and $I_m'(t)$ are determined by (20) and (21). The equations $I_R'(t)=0$ and $I_m'(t)=0$ lead to

$$bnR^{n-1}(t)\int_t^{\infty}\left[bd\int_0^{\tau}R^n(\xi)d\xi+B^d\right]^{1/d-1} m(\tau)\left[\frac{e^{-r\tau}-e^{-ra^{-1}(\tau)}}{r}-e^{-r\tau}k(\tau)\right]d\tau=e^{-rt}, \tag{27}$$

$$\int_t^{a^{-1}(t)}\left[1-\left[bd\int_0^{a(\tau)}R^n(\xi)d\xi+B^d\right]^{1/d}\Big/\left[bd\int_0^{t}R^n(\xi)d\xi+B^d\right]^{1/d}\right]e^{-r\tau}d\tau=k(t)e^{-rt} \tag{28}$$

at $t\in[t_l, \infty)$.

We will explore the possibility of exponential solutions for $R(t)$, while $m(t)$ and t-$a(t)$ are constant, to the system (26)-(28) separately in the cases $n=d$, $n>d$ and $n<d$. First of all, we start with the following preliminary result: If $R(t)$ is exponential, then $\beta(t)$ is *almost exponential* and practically undistinguishable from an exponent at large t in the sense of the following lemma:

Lemma 1. *If $R(t)=R_0e^{Ct}$ for some $C>0$, then*[10]

$$\beta(t) \approx R_0^{\,n/d}\left(\frac{bd}{Cn}\right)^{1/d} e^{Cnt/d} \tag{29}$$

at large t. In particular, $\beta(t)=R_0^{\,n/d}(bd/Cn)^{1/d}e^{Cnt/d}$ *if* $bdR_0^{\,n}=CnB^d$.

Proof. At $R(t)=R_0e^{Ct}$, $\beta(t)=\left(d\int_0^t bR_0^{\,n}e^{Cnv}dv+B^d\right)^{1/d}=\left(\frac{bdR_0^{\,n}}{Cn}e^{Cnt}-\frac{bdR_0^{\,n}}{Cn}+B^d\right)^{1/d}$.

Dividing $\beta(t)$ by $\tilde{\beta}(t)=R_0^{\,n/d}\left(\frac{bd}{Cn}\right)^{1/d}e^{Cnt/d}$, we obtain

$$\frac{\beta(t)}{\tilde{\beta}(t)}=\frac{1}{(bd/Cn)^{1/d}e^{Cnt/d}}\left(\frac{bd}{Cn}e^{Cnt}+\frac{B^d}{R_0^{\,n}}-\frac{bd}{Cn}\right)^{1/d}=\left[1+\left(\frac{CnB^d}{bdR_0^{\,n}}-1\right)e^{-Cnt}\right]^{1/d}$$

Expanding the function $(1+x)^c$ into the series, we obtain $\frac{\beta(t)}{\tilde{\beta}(t)}=1+\varepsilon(t)$, where the small parameter $\varepsilon(t)=\frac{1}{d}\left(\frac{CnB^d}{bdR_0^{\,n}}-1\right)e^{-Cnt}+\frac{1}{2d}\left(\frac{1}{d}-1\right)\left(\frac{CnB^d}{bdR_0^{\,n}}-1\right)^2 e^{-2Cnt}+\ldots$ decreases as e^{-Cnt}. The lemma is proven. □

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[10] For brevity, we will omit the expression "at large t" when using the notation $f(t)\approx g(t)$

We now define the concept of balanced growth paths considered.

Definition 1. *The Balanced Growth Path (BGP) is a solution (R, m, a) to (26), (27) and (28), where R is exponential and m(t) and t-a(t) are positive constants.*

If the environmental constraint (13) is not binding, then the system to be solved is

$$I_R'(t)=0, \qquad I_m'(t)=0, \qquad I_{a'}'(t)=0, \quad t\in[t_l\ ,\infty), \tag{30}$$

where $I_R'(t)$, $I_m'(t)$ and $I_{a'}'(t)$ are determined by (20), (24), and (25). As shown below, the optimal long-term growth with inactive regulation, $E<E_{\max}$, is possible only at $n>d$ (see Section 5.2.1).

Remark 5. In the case of the inactive environmental constraint (13), it is convenient to introduce the Freshet derivative

$$I_a'(t) = \rho(t)[p(t) - \beta(a(t))]m(a(t)), \tag{31}$$

of I in a instead of the Freshet derivative (25) in a' and use it during BGP analysis. Indeed, it is easy to see that if $I_{a'}'(t)\equiv 0$ at $t\in[\ t_l\ ,\infty)$ for some $t_l\geq 0$, then $I_a'(t)\equiv 0$ at $t\in[\ t_l\ ,\infty)$.

5.1. Balanced growth in case *n*=*d*.

In this case, the parameter of “R&D efficiency” n, $0<n<1$, is equal to the parameter of “R&D complexity” d, $0<d<1$. In this case, the optimal long-term growth involves the active environmental regulation at natural conditions. Namely,

Lemma 2. *At n=d, any interior solution (R, m, a) of the OP (10)-(17) with an exponentially growing R(t) involves the active environmental regulation:* $E(t)=E_{\max}(t)$ *starting at some* $t_l\geq 0$*, if the following conditions hold:*

(a) the environmental constraint $E_{max}(t)$ *is bounded on* $[0,\infty)$,

(b) the given energy price p(t) does not increase or increases slower than the optimal R(t): $\lim_{t\to\infty} p(t)/R(t) = 0$.

Proof. Let us consider $R(t)=R_0e^{Ct}$, then $\beta(t) \approx R_0(b/C)^{1/d}e^{Ct}$ by Lemma 1.

We assume that $E(t)<E_{\max}(t)$ at $[t_l,\infty)$, $t_l\geq 0$. Then, by Theorem 1, an OP interior regime (R, m, a) has to satisfy the nonlinear system (30). Substituting the above R and β into the expressions (20) and (31) for $I_R'(t)$ and $I_a'(t)$, we obtain from (30) that

$$R_0(b/C)^{1/d}e^{Ca(t)} = p(t), \tag{32}$$

$$bd(b/C)^{1-d}e^{C(d-1)t}\int_t^\infty e^{C(1-d)\tau}\left[\frac{e^{-r\tau}}{r}-\frac{e^{-ra^{-1}(\tau)}}{r}-e^{-r\tau}k(\tau)\right]m(\tau)d\tau \approx e^{-rt},\ \ t\in[t_l,\infty). \tag{33}$$

Equation (32) determines a, which is such that $t\text{-}a(t)\to\infty$ at $t\to\infty$ because of the condition (b). Equation (33) determines m at a given a. After introducing the function $f(t)$ = $[1-e^{-r[a^{-1}(t)-t]}]/r-k(t)$ and differentiating (33), we have

$$m(t) = [r-C(1-d)](C/b)^{1-d}/bd/f(t).$$

Since $f(t)<1/r$ for any possible k and a, then $m(t)>\text{const}=r[r-C(1-d)](C/b)^{1-d}/bd/r>0$. Therefore, by (13), $E(t)$ increases indefinitely at $t\to\infty$. At the condition (a), our assumption is wrong and $E(t)=E_{\max}(t)$ at some $t_1 > t_l$. Depending on the dynamics of the given $E_{\max}(t)$, it can become $E(t)<E_{\max}(t)$ at some $t_2 > t_1$, but $E(t)=E_{\max}(t)$ at $t\to\infty$. The lemma is proven. □

Though it makes use of a control variable (that is R(t)), the restriction (b) on the price of energy is actually quite natural . It will be refined along the way whence the optimal control R(t) better characterized. Indeed, as reflected in Section 6.1 below, when the energy price is too high (in the spirit of Lemma 2), the firm goes into a complete collapse with zero (optimal) investment in both equipment and R&D. Henceforth, energy consumption will itself go to zero, and the emission quota constraint will not be binding asymptotically. In this sense, energy prices play a role in the determination: too high prices will discourage any type of investment, leading the firm into a collapse in the long-run. This means that such a situation rather uncovers a case where the Hicksian mechanism does not work, which is hardly surprising: extreme input prices can never turn into investment incentives.

Condition (b) is sufficient. In Section 5.1.2, we will see a more specific a posteriori restriction (45) on the energy price increase, under which a balanced growth with the active environmental regulation takes place.

5.1.1. Growth under non-constant environment

The purpose of this section is to produce some natural "non-existence" results. In effect, one would expect that "too" stringent environmental regulation can by no means imply any counter-balancing Porter or induced-innovation mechanism. Our results confirm this intuition. On the other hand, as clearly highlighted in the 1975 Malcomson's work on the so-called Terborgh-Smith property related to the constancy of optimal scrapping time, for the model to generate regular exponential solutions with constant exponents, some strong conditions are needed on the forcing functions of the problem. Another insight gained by the analysis below is the role played by the price of capital equipment as reflected in the following theorem.

Theorem 2. *If $n=d$ and the conditions of Lemma 2 hold, then no exponential BGP with positive growth exists if the environmental constraint $E_{max}(t)$ monotonically decreases or increases OR the capital price $k(t)$ monotonically decreases or increases.*

Proof. By Lemma 2, $E(t)=E_{\max}(t)$ at $[t_l,\infty)$, $t_l\geq 0$. Then, by Theorem 1, an OP interior regime (R, m, a) has to satisfy the nonlinear system (26)-(28).

<u>Case 1:</u> $E_{\max}(t)$ decreases (increases). Then $m(t)=m(a(t))+dE_{\max}(t)/dt$ by (26), hence, $m(t)$ decreases (increases) from cycle to cycle and $m(t)$=const is not possible.

<u>Case 2:</u> $k(t)$ decreases (increases). Let us substitute $R(t)=R_0e^{Ct}$ into the FOC (27) and (28) for the active environment regulation case and estimate the obtained expressions at $t\to\infty$. Applying Lemma 1, we find that $\beta(t) \approx R_0(b/C)^{1/d}e^{Ct}$, then (27) leads to equation (33) and (28) leads to

$$\int_t^{a^{-1}(t)} \left[1-e^{C(a(\tau)-t)}\right] e^{-r\tau} d\tau \approx k(t)e^{-rt} \; . \tag{34}$$

The equality (34) determines $a(t)$ at the given $k(t)$. Its analysis shows that t-$a(t)$=const at $t\in[t_l,\infty)$, is possible only if $k(t)$=const, $t\in[t_l,\infty)$. If $k(t)$ decreases (increases), then t-$a(t)$ decreases (increases) and t-$a(t)\neq$const. The theorem is proven. □

It is interesting to compare the impact of the environmental constraint and capital price on long-term dynamics. By Theorem 2, if $E_{\max}(t)$ decreases and/or $k(t)$ increases over time, which cover the two unfavorable cases of increasingly stringent regulation and increasingly expensive capital goods respectively, then the BGP (in the sense of Definition 1) does not exist. In general, equation (33) demonstrates that the constraint

$$k(t) < 1/r, \tag{35}$$

is *necessary* for the existence of *any interior regime*. However, as follows from the proof of Theorem 2, the negative tendency (more stringent regulation) can be compensated (at a certain extent) with cheaper capital goods and, then, a long-term regime with an exponential $R(t)$ (and decreasing t-$a(t)$) is possible. Namely, the following refinement is interesting to report:

Corollary 1. *Let n=d and the environmental constraint (13) be active. Then:*

(a) If k(t) growths up to the value 1/r, then no interior regime with exponential growing R(t) is possible.

(b) If $E_{max}(t)\to 0$ at $t\to\infty$, then no such interior regime is possible.

(c) If both $E_{max}(t)$ and k(t) decrease at $t\to\infty$, then, interior regimes with exponential growing R(t) are possible in some ranges of E_{max} and k change. In particular, if k(t) monotonically decreases to 0 at $t\to\infty$, then $E_{max}(t)$ remains larger than a positive constant.

Proof continues Case 2 of the previous proof. Let us assume that the long-term $R(t)=R_0e^{Ct}$, $C>0$, and $m(t)$ is continuous. Substituting $R(t)$ into (26)-(28), we obtain (33) and (34). The statement (a) immediately follows from the positiveness of the left-hand part of (33).

Let $k(t)\to 0$ at $t\to\infty$. Then, it follows from (34) that the unknown t-$a(t)$ decreases and t-$a(t)\to 0$ at $t\to\infty$. By the mean value theorem, $E_{\max}(t)=[t-a(t)]m(\xi)$, where $a(t)<\xi<t$, and $E_{\max}(t)\approx[t-a(t)]m(a(t))$.

Next, one can see that (33) can hold only if

$$\left[\frac{1}{r}-\frac{e^{-r[a^{-1}(t)-t]}}{r}-k(t)\right]m(t)\approx C_E=\text{const}>0, \qquad (36)$$

where $C_E=[r-C(1-d)](C/b)^{1-d}/bd$ is found from (33). Since $a^{-1}(t)-t\to 0$ at $t\to\infty$, the function $f(t)=[1-e^{-r[a^{-1}(t)-t]}]/r-k(t)\to a^{-1}(t)-t-k(t)$ at $t\to\infty$. Therefore, (36) leads to $[a^{-1}(t)-t-k(t)]m(t)\approx C_E$ or $[t-a(t)-k(a(t))]m(a(t))\approx C_E$. Hence, $E_{\max}(t)\approx C_E+k(a(t))m(a(t))$

$>C_E>0$ and $E_{\max}(t)$ can not decrease to 0 (the statement (c)). If we assume $E_{\max}(t)\to 0$, then (36) can not be hold. It proves our statement (b). □

Corollary 1 demonstrates that the impact of the environmental constraint on the long-term optimal dynamics is more sensitive than the capital price. More specifically, an essential decrease in the environmental regulation $E_{\max}(t)$ cannot be compensated with the availability of cheaper capital goods (even if $k(t)\to 0$). No exponential growth is possible if $E_{\max}(t)\to 0$. However, exponentially decreasing capital prices are still compatible with exponentially rising R&D investment and decreasing finite nonzero quota emissions.

We now move to the case of constant economic and institutional environment, which is the case where balanced growth paths typically arise (see Malcomson, 1975).

5.1.2. Balanced growth under constant environment

The following theorem establishes the existence of balanced growth paths in the sense of Definition 1 when the economic and institutional environment is held constant.

Theorem 3 (*about **balanced growth***). *If*

$$n=d, \qquad E_{\max}(t)= E_0=\text{const}, \qquad k(t) = k =\text{const}, \tag{37}$$

then the interior optimal regime – BGP (R_Λ, m_Λ, a_Λ) exists,

$$R_\Lambda(t)\approx R_0e^{Ct}, \quad Q_\Lambda(t),\beta_\Lambda(t),c_\Lambda(t) \sim e^{Ct}, \quad m_\Lambda(t)=M_0=\text{const}, \quad a_\Lambda(t)=t- E_0/M_0, \tag{38}$$

where the constants C and M_0 are determined by the nonlinear system

$$C^{1/d}[r/C+d-1]=dM_0b^{1/d}\left[\frac{1-e^{-rE_0/M_0}}{r}-k\right], \tag{39}$$

$$\frac{1-e^{-rE_0/M_0}}{r}-\frac{e^{-CE_0/M_0}-e^{-rE_0/M_0}}{r-C}=k\ , \qquad (40)$$

that has a positive solution, at least, at small r. Namely, if r <<1 and

$$r^{1/d} < E_0 b^{1/d}[1-\sqrt{2kr}]\ , \qquad (41)$$

then C, 0<C<r, is a solution of the nonlinear equation

$$C^{(1-d)/d}[r-C(1-d)] = dE_0 b^{1/d}\left[1-\sqrt{\frac{k}{2}}\left(\frac{r}{\sqrt{C}}+\sqrt{C}\right)\right]+o(r) \qquad (42)$$

and $M_0 = E_0\sqrt{C/2k}+o(r)$. *The uniqueness of the solution is guaranteed if*

$$r^{1/d-1/2} < \frac{d^2}{4(1-d)}E_0 b^{1/d}\sqrt{2k}\ . \qquad (43)$$

If $\lim_{t\to\infty} p(t)e^{-Ct}=0$, *then the interior optimal* $c_\Lambda(t)>0$ *at large t.*

Proof. By Lemma 1, $\beta_\Lambda(t)\approx R_0(b/C)^{\frac{1}{d}}e^{Ct}$ at $n=d$. The substitution of (37), (38), and $\beta_\Lambda(t)$ into (27) leads to

$$bdM_0(R_0e^{Ct})^{n-1}\int_t^\infty\left(R_0\left(\frac{b}{C}\right)^{\frac{1}{d}}e^{C\tau}\right)^{1-n}\left[\frac{e^{-r\tau}-e^{-r(\tau+E_0/M_0)}}{r}-ke^{-r\tau}\right]d\tau = e^{-rt},$$

and, after integration, to

$$\frac{dM_0 b^{\frac{1}{d}}C^{\frac{n-1}{n}}}{C(1-d)-r}\left[\frac{kr-1}{r}+\frac{e^{-rE_0/M_0}}{r}\right]e^{-rt}=e^{-rt},$$

that can be rewritten as (39). Substituting (37), (38), and β_Λ to (28) gives

$$\int_t^{t+E_0/M_0}\left[1-e^{C(\tau-E_0/M_0)-Ct}\right]e^{-r\tau}d\tau \approx ke^{-rt},$$

which becomes (40) after integration.

Equations (39) and (40) may have a positive solution C and M_0 at natural assumptions. In particular, let r <<1. Then, presenting the exponents in (40) as the Taylor series, we obtain

$$\frac{1}{r}\left[1-1+rE_0/M_0-\frac{1}{2}(rE_0/M_0)^2+o(r^2)\right]$$
$$-\frac{1}{r-C}\left[1-CE_0/M_0+\frac{1}{2}(CE_0/M_0)^2+o(C^2)-1+rE_0/M_0-\frac{1}{2}(rE_0/M_0)^2+o(r^2)\right]=k$$

or

$$\left[E_0/M_0-\frac{r}{2}(E_0/M_0)^2\right]-\left[E_0/M_0-\frac{1}{2}\frac{1}{r-C}(r^2-C^2)(E_0/M_0)^2\right]+o(r)=k$$

or

$$(E_0/M_0)^2\left[(r+C)-r\right]+o(r)=2k\,,$$

which has the solution $M_0=E_0\sqrt{C/2k}+o(r)$.

Now, expressing the exponent in (39) as the Taylor series, we obtain

$$C^{(1-d)/d}[r-C(1-d)]=dM_0b^{1/d}\left[\frac{E_0}{M_0}-\frac{r}{2}\left(\frac{E_0}{M_0}\right)^2+o(r)-k\right] \qquad (44)$$

Substituting the obtained M_0 into (44) leads to one equation (42) for C. To analyze this equation, we use the new variable $x=\sqrt{C}$ and rewrite (42) as

$$F_1(x) = F_2(x),$$

where $F_1(x)=x^{2/d-2}(r+x^2(d-1))$, $F_2(x)=dE_0b^{1/d}\left[1-\sqrt{\frac{k}{2}}\left(\frac{r}{x}+x\right)\right]$

These functions are shown in Figure 1 and are such that $F_1(0)=0$, $F_1'(x)>0$ at $0<x<\sqrt{r}$, $F_1'(x)=0$ at $x=\sqrt{r}$, and $F_2'(x)>0$ at $0<x<\sqrt{r}$, $F_2'(x)=0$ at $x=\sqrt{r}$. Also, $F_2(x)<0<F_1(x)$ at small $0<x<<1$. Therefore, to have a solution $0 < x < \sqrt{r}$ to the latter equation, it is necessary and sufficient that $F_2(\sqrt{r})>F_1(\sqrt{r})$, which leads to the inequality (41). The sufficient condition for the uniqueness of x is $F_1'(x)<F_2'(x)$ at $0<x<\sqrt{r}$, which leads to (43).

Finally, let us prove that $c_A(t)>0$ at large t. By (12), $Q_\Lambda(t) \approx R_0 M_0 \left(\frac{b}{C}\right)^{1/d} \frac{1-e^{-CE_0/M_0}}{C} e^{Ct}$.

Therefore,

$$c_A(t) = Q_A(t) - k\beta_A(t)m_A(t) - R_A(t) - E_0 p(t)$$

$$\approx R_0 \left\{ M_0 \left(\frac{b}{C}\right)^{1/d} \left[\frac{1-e^{-CE_0/M_0}}{C} - k\right] - 1 \right\} e^{Ct} - E_0 p(t).$$

Expressing the exponent as the Taylor series and using (44), we obtain

$$\begin{aligned} c_\Lambda(t) &= R_0 \left\{ M_0 \left(\frac{b}{C}\right)^{1/d} \left[\frac{E_0}{M_0} - \frac{C}{2}\left(\frac{E_0}{M_0}\right)^2 + o(r) - k\right] - 1 \right\} e^{Ct} - E_0 p(t) \\ &> R_0 \left\{ \frac{r - C(1-d)}{Cd} - 1 \right\} e^{Ct} - E_0 p(t) = \frac{r-C}{Cd} R_0 e^{Ct} - E_0 p(t) \end{aligned} \tag{45}$$

If $\lim_{t\to\infty} p(t)e^{-Ct} = 0$, then $c_A(t)>0$ at large enough t for any positive value R_A.

The theorem is proven. □

The uniqueness condition (43) is sufficient. An analysis shows that the solution is usually unique without this condition. The only possible case of non-uniqueness (when we need this condition) is when the optimal C is very close to r. An additional analysis will be provided later to explore this issue. We now state some quite interesting comparative statics results highlighting how the balanced growth paths react to changes in the environmental regulation or price parameters.

Corollary 3. *At (37) and r<<1, a decrease of* E_0 *leads to the decrease of both optimal parameters C and* M_0*, and leave the long-term lifetime of capital goods unaltered since* $M_0 \sim E_0$*. A decrease of k decreases the optimal C, increases the optimal* M_0 *and diminishes the long-term lifetime of capital goods as* $M_0 \sim k^{-1/2}$.

More stringent environmental regulation through a decrease in E_0 is bad for the growth rate of firms' output and profit. Though firms respond to environmental regulation by exponential R&D investment efforts, the pace of such efforts is unambiguously negatively affected by increasingly stringent emission quotas. Strictly speaking, our long-term analysis rules out the occurrence of a kind of Porter hypothesis since a more stringent environmental regulation does reduce the growth rate of firms' output and profits. This extends the results of Xepapades and de Zeeuw (1999) and Feichtinger et al. (2005) in the missing direction. Even though the firms can respond to tighter emission quotas by more innovation, such an instrument does not allow to completely circumvent the impact of more severe regulation. In contrast, our result seems to go at odds with Hart's predictions (2004) according to which an emission tax may even boost the growth rate of production in the economy. However, the latter paper is based on a macroeconomic framework where the producers, and thus the polluters, are not entitled to spend on innovation. This might explain the difference, among other possible reasons.

Lower capital prices are good for investment (in energy consumptions units) but prove bad for the growth rate of firms' output and profit. This might sound as surprising. Nonetheless, one should keep in mind that lower capital prices may lead to declining R&D investment precisely because they tend to stimulate investment in physical capital, featuring a kind of substitution between investment in physical capital and investment in R&D. That is, the firm prefers to take profit from the exogenously decreasing capital prices rather than increasing its costly R&D expenditures. Since investment in R&D is the unique source of growth in this firm's problem, the growth rate may be penalized by decreasing capital price patterns. This is exactly what our model predicts in the long –run. A further interesting result concerns the optimal long-term lifetime of capital goods. Since $a_A(t)=t- E_0 /M_0$, and $M_0 \sim E_0$, it follows that a tighter environmental regulation leaves the optimal lifetime of capital goods unaltered. While a lower E_0 does reduce the optimal lifetime of machines, the fact that such a tighter regulation does also push investment downward pushes the maximizing firm to use the fewer machines longer. The two effects exactly offset each other in our framework. Under decreasing prices for capital goods, the firm invests more and uses the machines for a shorter time. This is somehow consistent with the recent literature on embodied technological progress observing that a more rapid investment-specific technological progress (like the one conveyed by the information and communication technologies) reduces the relative price of capital goods and decreases their lifetime due to rising obsolescence costs (see for example Krusell, 1998).

Remark 6. In Theorem 3, the BGP scale parameter R_0 appears to be undetermined. We have the indeterminacy of the long-term dynamics under the BGP because technical progress is endogenous. It happens for similar problems in the endogenous growth theory. A typical example is the Lucas-Uzawa model (see the book of Barro and Sala-i-Martin, Economic Growth, Chapter 5, and Boucekkine, Ruiz-Tamarit 2008).

Finally, it should be noted that the energy price p is not involved in the BGP (38). This is far from surprising since we have considered binding environmental constraints so far.[11] Indeed, under the active environmental constraint, the energy price p is not presented in the NCE formulas (18)-(22) and the optimal long term dynamics (R_A, m_A, a_A) will be the same for any p up to a certain level (that depends on the chosen indeterminacy parameter R_0). By Theorem 3, if the given energy price $p(t)$ increases slower than the optimal $R_A(t)$ (see also condition (b) of Lemma 2), then $c_A(t)\geq 0$ asymptotically.

The energy price p also has an indirect effect on the optimal controls R^*, a^*, m^*, since $p(t)$ impacts the endogenous $c^*(t)$: higher $p(t)$ means a lower level of cash $c^*(t)$, less money in the pocket. However, while the cash flow $c^*(t)$ is positive, the long-term firm optimal policy (BGP) is to invest the same in machines and in R&D.

We shall see in Section 6 that the energy price impacts the transition dynamics in our model, and it may be in such a way that we will never reach the BGP (see Section 6 below). The role of this price in the long-run dynamics is a valid question when $n>d$ (then the environmental regulation is not binding) and will be considered in Section 5.2 hereafter.

Remark 7. If equation (42) has a solution $0 < C < r$, then, in the general case, it has another solution C_2, $r<C_2<r/(1-d)$. However, the larger solution C_2 has no sense, since at $C>r$ the value of (1) is infinite and $c^*(t)<0$ by (45).

Numerical Example 1. Let r=0.05, d=0.5, b=0.01, E_0=10.5, and k=0.12. Then, the solution of the nonlinear system (38)-(39) is C=0.01 and $m_A(t)=M_0$ =2.1, which can be verified by its direct substitution into (38)-(39).

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[11] In such a case, the term $p(t)E(t)$ of the objective function becomes $p(t)E_{\max}(t)$, an exogenous term.

5.2. Cases *n<d* and *n>d*.

In these cases, no BGP in the sense of Definition 1 exists. However, a long-term regime with exponentially growing R and decreasing m appears to be possible at a special combination of given parameters.

Theorem 4. *Let $E_{max}(t)= E_0$=const and $k(t)= k$=const. Then:*

(a) If $n<d$, then no interior optimal regime with an exponentially growing R exists.

(b) If $n>d$, then an interior optimal regime (R_A, m_A, a_A) such that R_A grows exponentially, $m_A(t)\to 0$ at $t\to\infty$, and $E(t)<E_{max}(t)$, is possible ONLY if $p(t)\sim\exp(Cnt/d)$ where C is the endogenous rate of $R_A(t)$.

Proof. Let us substitute

$$R(t) = R_0e^{Ct} \qquad \text{and} \qquad m(t) = M_0e^{Dt}$$

into (27) and (28) and estimate the growth order of the obtained expressions at $t\to\infty$.

Applying Lemma 1 and Theorem 1, we find that $\beta(t)\approx R_0^{\;n/d}\left(\dfrac{bd}{Cn}\right)^{1/d} e^{Cnt/d}$,

$$I_R'(t)\approx bdR_0^{\;n-1}e^{C(n-1)t}\int_t^{\infty}\left[\frac{bd}{Cn}R_0^{\;n}e^{Cn\tau}\right]^{(1-d)/d}M_0\left[\frac{e^{-r\tau}}{r}-\frac{e^{-ra^{-1}(\tau)}}{r}-e^{-r\tau}k\right]e^{D\tau}d\tau-e^{-rt} \qquad (46)$$

If $E(t)$=$E_{\max}(t)$, then

$$I_m'(t)\approx\int_t^{a^{-1}(t)}\left[\beta(t)-\beta(a(\tau))\right]e^{-r\tau}d\tau-k\beta(t)e^{-rt} \qquad (47)$$

and the state variable $a(t)$ is found from (28) as

$$e^{-ra^{-1}(t)}=\left[\frac{D}{M_0}E(a^{-1}(t))+e^{Dt}\right]^{-r/D} \qquad (48)$$

Case ***n*<*d***: To keep $I_R'(t)=0$ by (46), we need an exponentially growing $m(t)$ with $D=C(1-n/d)>0$. Then $a(t)\to t$ by (48) and $I_m'(t)<0$ by (47), hence the optimal $m\equiv 0$. There is no interior regime with $C>0$ possible.

Case ***n*>*d***: By (46), the restriction

$$k<1/r \tag{49}$$

is necessary for $I_R'(t)\geq 0$. If (49) is valid, then, to keep $I_R'(t)=0$ by (46), we need an increasing $R(t)\sim e^{Ct}$ and a decreasing $m(t)\sim e^{Dt}$ with $D=C(1-n/d)<0$. The endogenous rate $C>0$ is to be determined.

Since $m(t)$ decreases exponentially, we have the case $E(t)<E_{\max}(t)$ with the inactive environmental regulation for any increasing $a(t)$. Then, by (24) and (31),

$$I_m'(t)=\int_t^{a^{-1}(t)}[\beta(t)-p(\tau)]e^{-r\tau}d\tau-\beta(t)ke^{-rt}=0\,, \tag{50}$$

$$I_a'(t)=\{p(t)-\beta(a(t))\}e^{-rt}m(a(t))=0 \tag{51}$$

The equation (51) has no solution $a(t)$ if $p\equiv$const. Let $p(t)=p_0e^{st}$, $s>0$. Then (51) has the form

$$I_a'(t)\approx\left\{p_0e^{st}-R_0^{\,n/d}\left(\frac{bd}{Cn}\right)^{1/d}e^{\frac{Cn}{d}a(t)}\right\}e^{-rt}m(a(t))=0. \tag{52}$$

Equation (52) has a unique solution a such that $a(t)\to(sd/Cn)t$. So, we have to assume $sd\leq Cn$ for keeping the constraint $a(t)<t$.

Let $sd<Cn$. Then $a(t)-t\to\infty$ by (52). Substituting $p(t)=p_0e^{st}$ into (50) , we obtain

$$I_m'(t)\approx\left(\frac{R_0^{\,n}bd}{Cn}\right)^{1/d}\left[\frac{e^{-(r-\frac{Cn}{d})t}}{r-\frac{Cn}{d}}-ke^{-(r-\frac{Cn}{d})t}\right]=\left(\frac{R_0^{\,n}bd}{Cn}\right)^{1/d}\frac{e^{-(r-\frac{Cn}{d})t}}{r-\frac{Cn}{d}}\left[1-rk+\frac{Cn}{d}k\right]>0\,,$$

for any C>0 because of (49). Therefore, no exponentially growing interior regime is possible in this case.

Finally, let $sd=Cn$. Then, by (52), $a(t)=t-L$, where L=const>0 at $p_0<B$. Substituting $a(t)$ and $p(t)=p_0e^{(Cn/d)t}$ into (46) and (50), we obtain a system of two nonlinear equations with respect to C and M_0. The system is similar to the equations (39)-(40) in Theorem 3. Its analysis shows the possibility of positive solutions C and M_0 at some restrictions on p_0 and other parameters. The theorem is proven. □

When $n>d$, the efficiency of the R&D investment appears to be higher as compared with the investment into the new capital. Theorem 4 concludes that, in the optimal long-time regime, almost all the output goes to R&D investment and the part of capital investments (exponentially) decreases in the total distribution of the output. Also, the *environment constraint is not binding* and we can keep a larger amount of older assets (since we buy an increasingly smaller amount of new capital).

By (49), the restriction $k(t)<1/r$ on the given capital price is necessary for the existence of any positive optimal regime. The energy price $p(t)$ plays an important role in the case $n>d$, in particular, an interior regime with an exponential R_A optimal path is impossible if the energy price $p(t)$ does not increase. Only if $p(t)$ increases with a certain rate, then an interior regime with exponentially increasing R_A and decreasing m_A is possible. The increase of $p(t)$ raises $a_A(t)$, that is, decreases the lifetime of capital goods. In other words, a kind of induced-innovation mechanism seems to be active in the case $n>d$, that is, when the R&D activity is highly efficient, so efficient that the investment devoted to equipment goes to zero. In such a case, the firm is in perpetually sharp modernization, and is not suffering at all from environmental regulation. We have to notice that this interior regime is not a BGP in the sense of Definition 1 because $m_A(t)$ asymptotically tends to zero. We shall disregard such a configuration in the short-term dynamics section below.

6. Transition dynamics

From now on, we set $n=d$. Since we have to deal with short-term dynamics in this section, some comments on the initial conditions are in order. The OP solution (R^*, m^*, a^*) satisfies the initial conditions (15). An essential initial condition is $a(0)=a_0$ because the unknown $a(t)$ is continuous. If $a_0 \neq a_A(0)$, then the dynamics of (R^*, m^*, a^*) involves a transition from the initial state $a(0)=a_0$ to the long term interior trajectory $a_A(t)$ (if it exists).

By (14), $c(0) = Q(0)-p(0)E(0)-R(0)-k(0)\beta(0)m_0(0) \geq 0$ at the initial state t=0, or

$$p(0)E(0)+k(0)Bm_0(0) \leq \int_{-a_0}^{0} \left[\beta_0 + \int_{-a_0}^{\tau} R_0(v)dv \right] m_0(\tau)d\tau - R_0(0) \tag{53}$$

(otherwise, the economic system is not possible at t=0 because of too high energy and capital prices $p(0)$, $k(0)$). Condition (59) implies two simpler constraints:

$$p(0) < B_0 \qquad \text{and} \qquad k(0)m_0(0) < E(0). \tag{54}$$

Even if (53) holds, the optimal dynamics may be such that the economic system will never reach the environmental restriction $E(t)=E_0(t)$ because of too high energy and/or capital prices. Let us demonstrate the corresponding scenarios.

6.1. The collapse cases.

Let $E(0)<E_{\max}(0)$ at the initial time .

Scenario 1: ***The case of too high energy price.*** Let us assume that the external market energy price $p(t)$ increases faster than the optimal $\beta(t)$ (e.g., if β is an exponent with the rate larger than C in the case $d=n$). Then, by (25), $I_{a'}'(t)>0$ for all t and the optimal strategy is to keep the lifetime of the capital $t-a^*(t)$ as short as possible because of the high energy cost $p(t)$. In this case, the optimal $a^*(t)$ soon becomes $a^*(t)=t$ and the optimal new investment $m^*(t)=0$ is determined by the sign $I_m'(t)<0$ in (24). By (20), $I_R'(t)<0$ and the optimal $R^*(t)=0$. So, the optimal dynamics is a situation of ***an economic collapse*** (no capital renovation and complete scrapping of existing capital) because of too high price of the resource. By (13), the variable $E(t)$ strives to 0 and is always less than $E_{\max}(t)$. If $p(t)$ increases with exactly the same rate as the optimal $\beta(t)$, then the economy may grow but never reach the environmental constraint $E(t)=E_{\max}(t)$.

Scenario 2: ***The case of too high capital cost.*** Let the market price $k(t)$ of new capital increases (not even indefinitely) and becomes $k(t)\geq 1/r$ at $t\geq t_{cr}$ for some $t_{cr}>0$. Then, by (24), $I_m'(t)<0$ and the optimal new investment $m^*(t)=0$ at $t\geq t_{cr}$. So, the optimal strategy is to buy no new capital. By (20), $I_R'(t)<0$ and the optimal $R^*(t)=0$. By (25), $I_{a'}'(t)<0$ and, hence, $a^*(t)=a_0$. The optimal dynamics is the ***economic decline*** (no R&D investment, no capital renovation and no capital scrapping) because of too high price $k(t)$ of the new capital. The variable $E(t)\leq E(0)<E_{\max}(0)=E_{\max}(t)$, i.e., the environmental constraint is never reached.

The above scenarios do not reflect the nature of technological capital replacement. In Section 6.2 below, we consider cases when the optimal system dynamics involves capital renovation.

The OP produces qualitatively different optimal regimes $R^*(t)$, $m^*(t)$, $a^*(t)$, $t\in[0,\infty)$, depending on whether the environmental balance restriction (13) is active, $E(t)=E_{max}(t)$, or inactive, $E(t)<E_{max}(t)$. In our model, the firms-polluters are the firms for which the restriction is active. We will consider the cases of firms-polluters and firms-non-polluters separately. At $n=d$, the long-term BGP dynamics involves the active environmental restriction (13) (see Theorem 3). As shown below, the transition dynamics reaches the restriction (13).

6.2. Optimal intensive growth (the case of a dirty firm).

Let us assume that $E(t_k)=E_{max}(t_k)$ starting with the instant t_k, $t_k\geq 0$. Also $k(0)<1/r$, otherwise no growth is possible (see Scenario 2 above).

Scenario 3: ***The intensive growth at active environment regulation.*** Let $t_k=0$. The optimal dynamics at $t\geq t_k$ follows Case A of Theorem 1 (with the active $E(t)=E_{max}(t)$ restriction). This regime is *a* ***growth with intensive capital renovation*** induced by technical progress. In order to make a new capital investment $m(t)$ at $t\geq t_k$, the firm needs to scrap some obsolete capital $m(a(t))a'(t)$, following equality (13) under the given $E(t)=E_{max}(t)$. In the long-term dynamics considered in Section 3.2, the optimal R&D innovation $R^*(t)$ is the interior trajectory $R_\Lambda(t)$. The optimal $R^*(t)$ reaches the trajectory $R_\Lambda(t)$ immediately after t_k. The OP has the interior turnpike trajectory a_Λ for the capital lifetime, determined from $I_m'(t)=0$. If $a_A(0)=a_0$, then the optimal $a^*\equiv a_A$. If $a_A(0)\neq a_0$, then we can show that the optimal $a^*(t)$ will reach $a_A(t)$ after some time at $t_l\geq t_k$. If $a_A(0)<a_0$, then the optimal investment $m^*(t)=0$ is minimal at $0<t\leq t_l$. Later, at $t>t_l$, the optimal $m^*(t)$ has cycles (the *replacement echoes* as in Boucekkine, Germain and Licandro, 1997) determined by the prehistory of $m(t)$ on $[a_0, t_k]$. A formal proof of this optimal m^*, a^* dynamics can be done similarly to Hritonenko and Yatsenko (2005).

Under Scenario 3, the energy price $p(t)$ is not presented in the NCE formulas (18)-(22) and, therefore, it does not impact the optimal transition $a^*(t)$ and $m^*(t)$ (similarly to the long term dynamics) while restrictions (45) and (53) hold. Of course, an increase of the energy price $p(t)$ reduces the corresponding optimal $Q^*(t)$ and $c^*(t)$.

Figure 2 and the following simulation example illustrate this scenario.

Example 2. Let

$$r=0.05,\quad d=n=0.5,\quad b=0.01,\quad E_{\max}(t)=E_0=10.5,\quad k(t)=0.12,\quad p(t)=0.5,$$
$$a_0=-2,\quad \beta_0=1,\quad R_0(\tau)=0,\quad m_0(\tau)=5.25,\quad \tau\in[-2,0]. \tag{55}$$

Then, $B=\beta(0)=1$ by (16). There is the BGP, calculated in Example 1 above,

$$R_A(t)=R_0e^{Ct},\ C=0.01,\quad m_A(t)=M_0=2.1,\quad a_A(t)=t-5\ ,\ t\in[0,\infty), \tag{56}$$

indicated by the grey lines in Figure 2. In this case, $E(0)=m_0a_0=5.25*2=10.5$ is equal to $E_{\max}(0)$, hence, the environmental balance (13) is active starting $t=0$. Since $a_A(0)=-5<a_0=-2$, then the optimal $a^*(t)=-2$ and $m^*(t)=0$ at $0<t\leq t_l=3$. After t_l, the optimal $a^*(t)$ coincides with the BGP $a_A(t)$ and $m^*(t)=m^*(t\text{-}5)$ exhibits *replacement echoes*.

6.3. Optimal extensive growth (the case of a firm-non-polluter).

This case means that the energy pollution balance $E(t)$ at the initial state $t=0$ is lower than the limit $E_{\max}(0)$. Let us assume that $E(t)<E_{\max}(t)$ at $0\leq t<t_k$, where the moment t_k will be

determined. Mathematically, this case is more complicated and involves the regime $c^*(t)=0$, not covered by the NCE of Theorem 1. So, we restrict ourselves with a numeric example and its economic interpretation and do not provide any formal proofs.

Scenario 4: ***Extensive growth.*** If $p(t)$ does not increase, then by (25), (31), and (54) $I_{a'}'(t)<0$ and $I_a'(t)<0$ at $a(t)\equiv a_0$. Hence, $a^*(t)\equiv a_0$ is optimal while $E(t)<E_{\max}(t)$ (see Example 3 below). At the same time, if $\beta^*(t)$ increases (and the given $k(t)$ does not increase), then $I_m'(t)>0$ by (24), hence, the optimal investment m^* is maximal. So, one can buy a new capital and there is no need to remove the old one, i.e., we have ***an extensive growth***. On the other side, by (20), $I_R'(t)>0$ at small $R^*(t)$, hence, the optimal $R^*(t)$ is positive (therefore, $\beta^*(t)$ increases indeed). In this case, the constraint $c^*(t)\geq 0$ in (14) limits both controls R^* and m^*:

$$Q^*(t) - R^*(t) - k(t)\beta^*(t)m^*(t) - p(t)E^*(t) \geq 0. \tag{57}$$

Then, the transition optimal dynamics on some initial period $[0, t_k]$ is determined by the restriction $c^*(t)=0$ or

$$R^*(t) + k(t)\beta^*(t)m^*(t) = Q^*(t) - p(t)E^*(t) \tag{58}$$

until $E(t_k)=E_{\max}(t_k)$. Since the optimal m^* increases, the energy regulation limit $E(t)=E_{\max}(t)$ will be reached soon and the optimal renovation dynamics will switch to Scenario 3 with the active constraint (13). The end t_k of the initial transition period $[0, t_k]$ is determined from condition $E(t_k)= E_{\max}(t_k)$.

If the given $p(t)$ increases indefinitely, then $a^*(t)$ is determined by (35), $I_{a'}'(t)=0$, $I_a'(t)=0$, $\beta(a^*(t))=p(t)$, hence, $a^*(t)$ increases. If the $p(t)$ increase is slower than the optimal productivity, then the optimal capital lifetime $t-a^*(t)$ also increases while $E(t)<E_{\max}(t)$.

Example 3. Let all given parameters be as (55) in Example 2 but

$$m_0(\tau) = 2, \quad \tau \in [-2,0]. \tag{59}$$

Then the BGP (56) is the same as in Example 2 but the transition dynamics is different.
In this case, $E(0)=m_0a_0=2*2=4$ is less than $E_{\max}(0)=10.5$, hence, the environmental balance (13) is inactive on an initial interval $[0, t_k]$ at the beginning of the planning horizon. The dynamics of the optimal $m^*(t)$ and $R^*(t)$ on $[0, t_k]$ follows the restriction $c^*(t)=0$ and is shown in Figure 3. The determination of m^* and R^* involves additional theoretical considerations based on varying the equality (58). It appears that $m^*(t)=17.8$, $R^*(t)=0.003$ at $0\le t\le t_k$. Then, the corresponding $E^*(t)$ increases fast and reaches the limit value $E_{\max}=10.5$ at $t_k\approx 0.36$. The later optimal dynamics on $[t_k,\infty)$ is described by Case A of Theorem 1 and is similar to Scenario 3. Namely, since $a_A(0.36)<a_0=-2$, then $a^*(t)=-2$ and the optimal $m^*(t)=0$ is minimal during the second part of transition dynamics, $0.36 < t \le t_l=3$. Later, at $t>3$, $a^*(t)=a_A(t)$ and the optimal $m^*(t)=m^*(t-5)$ has *replacement echoes* determined by the previous dynamics on [-2, 3].

Remark 8. If the positiveness of $c(t)$ is not assumed, then the optimal $m^*(t)$ jumps to infinity immediately after $t=0$ (because of the possibility of borrowing), so the balance $E(t)=E_{\max}(t)$ will be reached immediately after $t=0$. Mathematically, $m^*(t)$ involves the delta-function at $t=0$. Then the length of the transition period $[0, t_k]$ is zero.

The optimal dynamics highlighted in this scenario are quite new in the related economic literature (see for example, Boucekkine, Germain and Licandro, 1997). They deserve some comments:

i) At first, note that the modernization policy chosen by the firm consists in increasing investment in new equipment and R&D **without** scrapping the older and more polluting machines. In Hritonenko and Yatsenko (1996) and Boucekkine et al. (1997), the modernization policy also encompasses scrapping part of the older capital goods in a way similar to the intensive growth scenario described in Section 6.2. The reason behind this difference is quite elementary:

while in Section 6.2, investing in new machines (for fixed level of technological progress) is not possible without scrapping some obsolete older machines because of market clearing conditions or binding environmental constraints respectively, a firm with low enough initial capital stock (and so with low enough initial pollution stock) has no incentive to scrap its old machines as long as its emission quota constraint is not binding.

ii) Note that in our case firms which are historically "small" polluters are precisely those which are historically "small" producers. Extended to a country level, our exercise predicts that historically poor countries will find it optimal to massively invest and therefore to massively pollute during their development process. During such a transition, new and clean machines will co-exist with old and dirty machines in the productive sectors, implying an unambiguously dirty transition. In this sense, our model provides new micro-foundations to an essential part of the environmental Kuznets curve (see among others, Chimeli and Braden, 2005).

The next section establishes that during this transition, the unique brake on pollution is the energy price, which suggests a fiscal treatment of the environmental problem during the transition. Nonetheless, the trade-off is clear: if energy taxes are raised to cut energy consumption, then it will affect the pace of technological progress and investment negatively, featuring a kind of reverse Hicks mechanisms, as proved in the next section.

6.4. The impact of energy price on extensive growth.

In the case of extensive growth (Scenario 4), the transition dynamics is directly impacted by the behavior of the energy price $p(t)$ because of (25).

Let $p(t)$ monotonically increase. Then $a^*(t)$ increases and is uniquely determined from $I_a'(t)=0$ and (25) at a known R^*, while $E^*(t)< E_{\max}(t)$. The long term interior trajectory $a_A(t)$ defined in Section 5.1.2 satisfies the equation $I_m'(a_A; t)=0$. We assume that $p(t)$ is not too high, so that $I_m'(a^*; t)>0$ in (21) and $m^*(t)$ is maximal during transition dynamics (the alternative case is Scenario 1 in Section 6.1). Then, the transition dynamics is regime $c^*(t)=0$ and $m^*(t)$ and $R^*(t)$ satisfy (58) while $E^*(t)<E_{\max}(t)$ on an initial interval $[0, t_k]$.

We will compare the transition dynamics under two different (increasing) energy prices $p_1(t)$ and $p_2(t)$, $p_1(t) < p_2(t)$, and indicate corresponding optimal $a^*(t)$, $m^*(t)$, and $R^*(t)$ with the subscripts $_1$ and $_2$. By (25), the structure of the equation $I_a'(t)=0$ is such that $a_1^*(t)<a_2^*(t)$ while $E^*(t)<E_{\max}(t)$. The endogenous Q^*, β^*, and E^* are more inertial than m^* and R^*. On the other hand, the "extensive growth" part $[0, t_k]$ of transition dynamics is usually very short (see Example 3). Then, by (58),

$$R_1^*(t) + k(t)\beta^*(t)m_1^*(t) > R_2^*(t) + k(t)\beta^*(t)m_2^*(t) \tag{60}$$

at $[0, t_k]$. Involving additional reasoning based on varying the equality (58), we can prove that $R_1^*(t)>R_2^*(t)$ and $m_1^*(t)>m_2^*(t)$. Therefore, both $R^*(t)$ and $m^*(t)$ are smaller at a higher energy price $p(t)$. This result can be summarized as the following property:

During the transition dynamics with inactive environmental constraint, an increase of the energy price p(t) forces more intensive capital renovation with a shorter capital lifetime t-a(t) but decreases both capital and R&D investments R*(t) and m*(t).*

Let us highlight the extreme case of the prices $p_1(t)$ and $p_2(t)$ such that $Q(0)-p_1(0)E(0)>0$ but $Q(0)-p_2(0)E(0)=0$. Then, by (58), $R^*_1(0)$ and (or) $m^*_1(0)$ are positive, but $R^*_2(0)=m^*_2(0)=0$ since all given output $Q(0)$ is spent at t=0 to buy energy because of too high energy price. Under natural assumptions, the production will never become profitable at the price $p_2(t)$.

As outlined above, we get here a case for an inverse induced-innovation mechanism (under inactive environmental constraint). Higher energy prices induce shorter lifetime for capital goods but they depress investment in both new capital and R&D.

7. Concluding remarks

In this paper, we have studied in depth the optimal behavior of firms subject to emission quotas and liquidity-constrained. We have spent a substantial part of the first sections of the paper to justify why such a problem under endogenous technical progress (that is, when firms spend on R&D) is crucially important to tackle. In addition, the vintage structure adds realism to the problem under study and considerably enriches the discussion. We have extracted numerous new results, either in the investigation of short-term dynamics or in the analysis of long-run growth regimes. In most cases analyzed, the Porter and induced-innovation hypotheses are ruled out.

A few remarks are in order. Of course, our results are based on price-taking firms and our modeling of liquidity-constraints is probably too simple. Adding market power is no problem if we follow the strategy of Feichtinger et al. (2006), although it is not likely that our results would be dramatically altered. Modelling and treating liquidity constraints more accurately is a much more complicated task both mathematically and conceptually.

We believe that allowing the firms to incur into debt to fasten its modernization and compliance to environmental standards is a quite decisive issue that should be considered in more comprehensive frameworks in the terms of economic policy. In this spirit, central planner models seem more adequate, since they would allow a much more precise discussion of welfare implications of different environmental and economic policies. This is our next step.

8. Appendix

Proof of Theorem 1: The proof uses perturbation techniques of the optimization theory developed for the class of models under study in Hritonenko and Yatsenko (1996), Yatsenko (2004), and Yatsenko and Hritonenko (2005). Let us consider Case (B) first.

Case (B). If the restriction (13) is inactive, $E^*(t)<E_{\max}(t)$ at $t\in\Delta$, then we choose R, m, and $v=a'$ as the *independent* unknown variables of the OP. Then, the differential restriction $a'(t)\geq 0$ in (14) has the standard form $v(t)\geq 0$. We assume that R, m, and v are measurable and $R(t)e^{-rt}$, $m(t)e^{-rt}$, $v(t)e^{-rt}$ are bounded *a.e.* on $[0,\infty)$. Substituting (17) to (16), we obtain expression (22) for $\beta(t)$.

We refer to measurable functions δR, δm, and δv as the *admissible variations*, if R, m, v, $R+\delta R$, $m+\delta m$, and $v+\delta v$, satisfy constraints (14)-(15).

Let us give small admissible variations $\delta R(t)$, $\delta m(t)$, and $\delta v(t)$, $t\in[0,\infty)$, to a, m, and R and find the corresponding variation $\delta I = I(R+\delta R, m+\delta m, v+\delta v) - I(R,m,v)$ of the objective functional I. Using (10)-(13), we obtain that

$$
\begin{aligned}
\delta I = \int_0^\infty e^{-rt}[&\int_{a(t)+\delta a(t)}^{t} \left(db\int_0^\tau (R(\xi)+\delta R(\xi))^n d\xi + B^d \right)^{\frac{1}{d}} (m(\tau)+\delta m(\tau))d\tau \\
&- p(t) \int_{a(t)+\delta a(t)}^{t} (m(\tau)+\delta m(\tau))d\tau - (R(t)+\delta R(t)) \\
&- k(t)(m(t)+\delta m(t)) \left(db\int_0^t (R(\xi)+\delta R(\xi))^n d\xi + B^d \right)^{\frac{1}{d}}]dt \qquad \text{(A1)} \\
- \int_0^\infty e^{-rt}[&\int_{a(t)}^{t} \left(db\int_0^\tau R^n(\xi)d\xi + B^d \right)^{\frac{1}{d}} m(\tau)d\tau - p(t)\int_{a(t)}^{t} m(\tau)d\tau - R(t) \\
&- k(t)m(t) \left(db\int_0^t R^n(\xi)d\xi + B^d \right)^{\frac{1}{d}}]dt
\end{aligned}
$$

where $\delta a(t) = \int_0^t \delta v(\xi)d\xi$. To prove the Theorem, we shall transform the expression (A1) to the form

$$
\delta I = \int_0^\infty (I'_R(t)\cdot \delta R(t) + I'_m(t)\cdot \delta m(t) + I'_v(t)\cdot \delta v(t))dt + o(\|\delta R\|, \|\delta m\|, \|\delta v\|) , \qquad \text{(A2)}
$$

where the norm is $\|f\| = \underset{[0,\infty)}{ess\sup} |e^{-rt} f(t)|$. It will involve several steps. First, using the Taylor expansion $f(x+\delta x)=f(x)+f'(x)\delta x+o(\delta x)$ twice, we have that

$$
\begin{aligned}
&\left(db\int_0^\tau (R(\xi)+\delta R(\xi))^n d\xi + B^d \right)^{\frac{1}{d}} \\
&= \left(db\int_0^\tau (R^n(\xi)+nR^{n-1}(\xi)\delta R(\xi)+o(\delta R(\xi)))^n d\xi + B^d \right)^{\frac{1}{d}} \qquad \text{(A3)} \\
&= \beta(\tau) + bn\beta^{1-d}(\tau)\int_0^\tau R^{n-1}(\xi)\delta R(\xi)d\xi + \int_0^\tau o(\delta R(\xi))d\xi.
\end{aligned}
$$

Next, using (A3) and the elementary property $\int_{a(t)+\delta a(t)}^{t} f(\tau)d\tau = \int_{a(t)}^{t} f(\tau)d\tau - \int_{a(t)}^{a(t)+\delta a(t)} f(\tau)d\tau$ of integrals, we transform (A1) to

$$\delta I = \int_0^\infty e^{-rt}[bn \int_{\max\{a(t),0\}}^{t} m(\tau)\beta^{1-d}(\tau)\int_0^\tau R^{n-1}(\xi)\delta R(\xi)d\xi d\tau dt$$
$$+\int_0^\infty e^{-rt} \int_{\max\{a(t),0\}}^{t} (\beta(\tau)-p(t))\delta m(\tau)d\tau dt$$
$$+\int_0^\infty e^{-rt} \int_{a(t)}^{a(t)+\delta a(t)} (p(t)-\beta(\tau))m(\tau)d\tau dt - \int_0^\infty e^{-rt}[\delta R(t)+k(t)\beta(t)\delta m(t)]dt \tag{A4}$$
$$-\int_0^\infty e^{-rt}k(t)m(t)\beta^{1-d}(t)\int_0^t bnR^{n-1}(\xi)\delta R(\xi)d\xi dt + \int_0^\infty e^{-rt}o(\delta R(t),\delta m(t))dt,$$

where $\max\{a(t),0\}$ emphasizes that the variations $\delta R(t)$, $\delta m(t)$ are non-zero only on the interval $[0,\infty)$.

Next, we interchange the limits of integration in the second term of (A4) as

$$\int_0^\infty e^{-rt}[\int_{a(t)}^{t} (\beta(\tau)-p(t))\delta m(\tau)d\tau dt = \int_0^\infty \int_t^{a^{-1}(t)} e^{-r\tau}(\beta(t)-p(\tau))d\tau\cdot\delta m(t)dt,$$

in the first term as

$$\int_0^\infty e^{-rt}[\int_{\max\{a(t),0\}}^{t} m(\tau)\beta^{1-d}(\tau)\int_0^\tau bnR^{n-1}(\xi)\delta R(\xi)d\xi d\tau]dt$$
$$= bn\int_0^\infty \int_t^\infty \int_\tau^{a^{-1}(\tau)} e^{-r\xi}d\xi\cdot m(\tau)\beta(\tau)^{1-d}d\tau\cdot R^{n-1}(t)\delta R(t)dt,$$

and in the fifth term similarly. To transform the third term, we use the Taylor expansion $\int_{a(t)}^{a(t)+\delta a(t)} f(t,\tau)d\tau = f(t,a(t)) + o(\delta a(t))$. Collecting coefficients of δR, δm, and δa, we rewrite (A4) as:

$$\delta I = \int_0^\infty [-e^{-rr} + bn\int_t^\infty (\int_\tau^{a^{-1}(\tau)} e^{-r\xi}d\xi - e^{-r\tau}k(\tau))\cdot m(\tau)\beta^{1-d}(\tau)d\tau\cdot R^{n-1}(t)]\cdot\delta R(t)dt$$
$$+\int_0^\infty [\int_t^{a^{-1}(t)} e^{-r\tau}(\beta(t)-p(\tau))d\tau - e^{-rt}k(t)\beta(t)]\cdot\delta m(t)dt$$
$$+\int_0^\infty e^{-rt}(p(t)-\beta(a(t))m(a(t))\cdot\delta a(t)dt + \int_0^\infty e^{-rt}o(\delta R(t),\delta m(t),\delta a(t))dt.$$

Finally, recalling $\delta a(t)=\int_0^t \delta v(\xi)d\xi$, we convert the last expression to

$$
\begin{aligned}
\delta I = &\int_0^\infty [-e^{-rr} + bn\int_t^\infty (\int_\tau^{a^{-1}(\tau)} e^{-r\xi}d\xi - e^{-r\tau}k(\tau))\cdot m(\tau)\beta^{1-d}(\tau)d\tau \cdot R^{n-1}(t)]\cdot \delta R(t)dt \\
&+\int_0^\infty [\int_t^{a^{-1}(t)} e^{-r\tau}(\beta(t)-p(\tau))d\tau - e^{-rt}k(t)\beta(t)]\cdot \delta m(t)dt \qquad \text{(A5)} \\
&+\int_0^\infty \int_t^\infty e^{-r\tau}(p(\tau)-\beta(a(\tau))m(a(\tau))d\tau \cdot \delta v(t)dt + \int_0^\infty e^{-rt}o(\delta R(t),\delta m(t),\delta v(t))dt
\end{aligned}
$$

Formula (A5) in notations (21), (24), (25) provides the required expression (A2). The domain (14) of admissible controls R, m, v has the simple standard form $R\geq 0$, $m\geq 0$, $v\geq 0$. So, the NCE (23) follows from the obvious necessary condition that the variation δI of functional I can not be positive for any admissible variations $\delta R(t)$, $\delta m(t)$, $\delta v(t)$, $t\in[0,\infty)$.

Case (A). If the restriction of (13) is active: $E(t) = E_{\max}(t)$ at $t\in\Delta\subset[0,\infty)$, then we choose R and m as the *independent unknowns* of the OP. The dependent (*state*) variable a is uniquely determined from the initial problem

$$m(a(t))a'(t) = m(t) - E_{\max}'(t), \quad a(0)= a_0,$$

obtained after differentiating (13). As shown in Hritonenko and Yatsenko (2006), if $E_{\max}'(t)\leq 0$, then for any measurable $m(t)\geq 0$, a unique a.e. continuous function $a(t)<t$ exists and a.e. has $a'(t)\geq 0$ (see Remark 1 about the possible case $E_{\max}'(t)>0$). Therefore, the state restrictions $a'(t)\geq 0$ and $a(t)<t$ in (14) are satisfied automatically, so we can exclude a from the extremum condition.

Similarly to the previous case, let us give small admissible variations $\delta R(t)$ and $\delta m(t)$, $t\in[0,\infty)$, to R and m and find the corresponding variation $\delta I = I(R+\delta R, m+\delta m) - I(R,m)$

of the functional *I*. In this case, the variation δa is determined by δm. To find their connection, let us present (13) as

$$E_{\max}(t) = \int_{a(t)}^{t} m(\tau)d\tau = \int_{a(t)+\delta a(t)}^{t} (m(\tau)+\delta m(\tau))d\tau$$

then

$$\int_{\max\{a(t),0\}}^{t} \delta m(\tau)d\tau = \int_{a(t)}^{a(t)+\delta a(t)} m(\tau)d\tau + o(\|\delta m\|, \|\delta a\|)\ . \qquad \text{(A6)}$$

We will use the above formula (A4) for the variation δI as a function of δR, δm, and δa and eliminate δa from (A4) using (A6). To do that, we rewrite the third term of (D4) as

$$\begin{aligned}
&\int_0^\infty e^{-rt} \int_{a(t)}^{a(t)+\delta a(t)} (p(t)-\beta(\tau))m(\tau)d\tau dt \\
&= \int_0^\infty e^{-rt}(p(t)-\beta(a(t))) \int_{a(t)}^{a(t)+\delta a(t)} m(\tau)d\tau dt + \int_0^\infty e^{-rt} \int_{a(t)}^{a(t)+\delta a(t)} (\beta(a(t))-\beta(\tau))m(\tau)d\tau dt \\
&= \int_0^\infty e^{-rt}(p(t)-\beta(a(t)) \int_{\max\{a(t),0\}}^{t} \delta m(\tau)d\tau dt + \int_0^\infty e^{-rt} o(\delta a(t), \delta m(t))dt \\
&= \int_0^\infty \int_t^{a^{-1}(\tau)} e^{-r\tau}(p(\tau)-\beta(a(\tau))d\tau \cdot \delta m(t)dt + \int_0^\infty e^{-rt} o(\delta a(t), \delta m(t))dt
\end{aligned} \qquad \text{(A7)}$$

by adding $\pm\int_0^\infty e^{-rt}\beta(a(t)) \int_{a(t)}^{a(t)+\delta a(t)} m(\tau)d\tau dt$ and applying (A6). The integral

$\int_a^{a+\delta a} (\beta(a)-\beta(\tau))m(\tau)d\tau$ in (A7) has the order $o(\delta a)$ because $\beta(\tau)$ is continuous.

Substituting (A7) into (A4) and collecting the coefficients of δm and δR, we obtain the expression

$$\delta I = \int_0^\infty (I'_R(t)\cdot \delta R(t) + I'_m(t)\cdot \delta m(t))dt + o(\|\delta R\|, \|\delta m\|) \qquad \text{(A8)}$$

in the notations (20) and (21). The rest of the proof is identical to Case B.

The Theorem is proven. □

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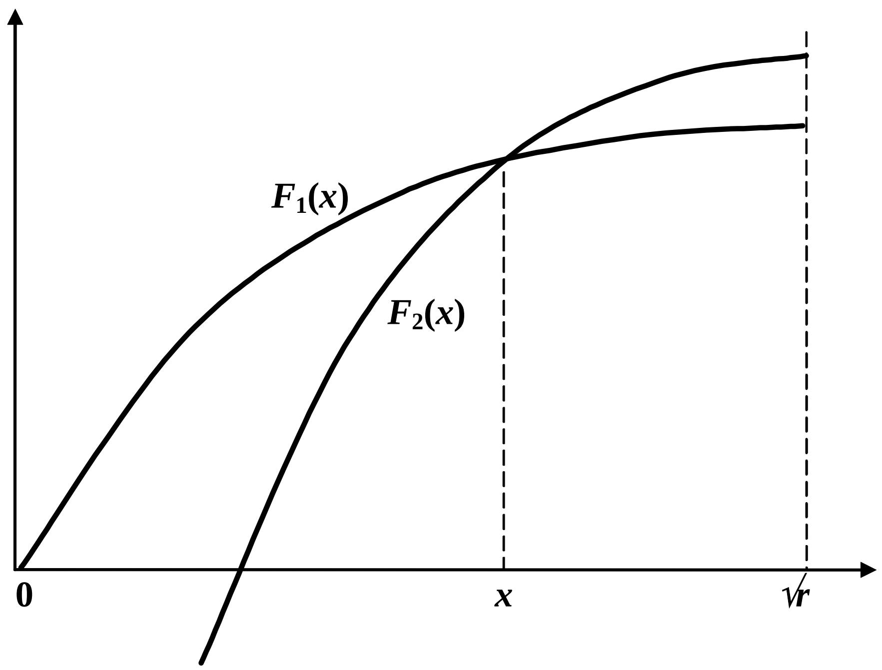

Figure 1. Solving the nonlinear equation (45) with respect to the unknown $x = \sqrt{C}$.

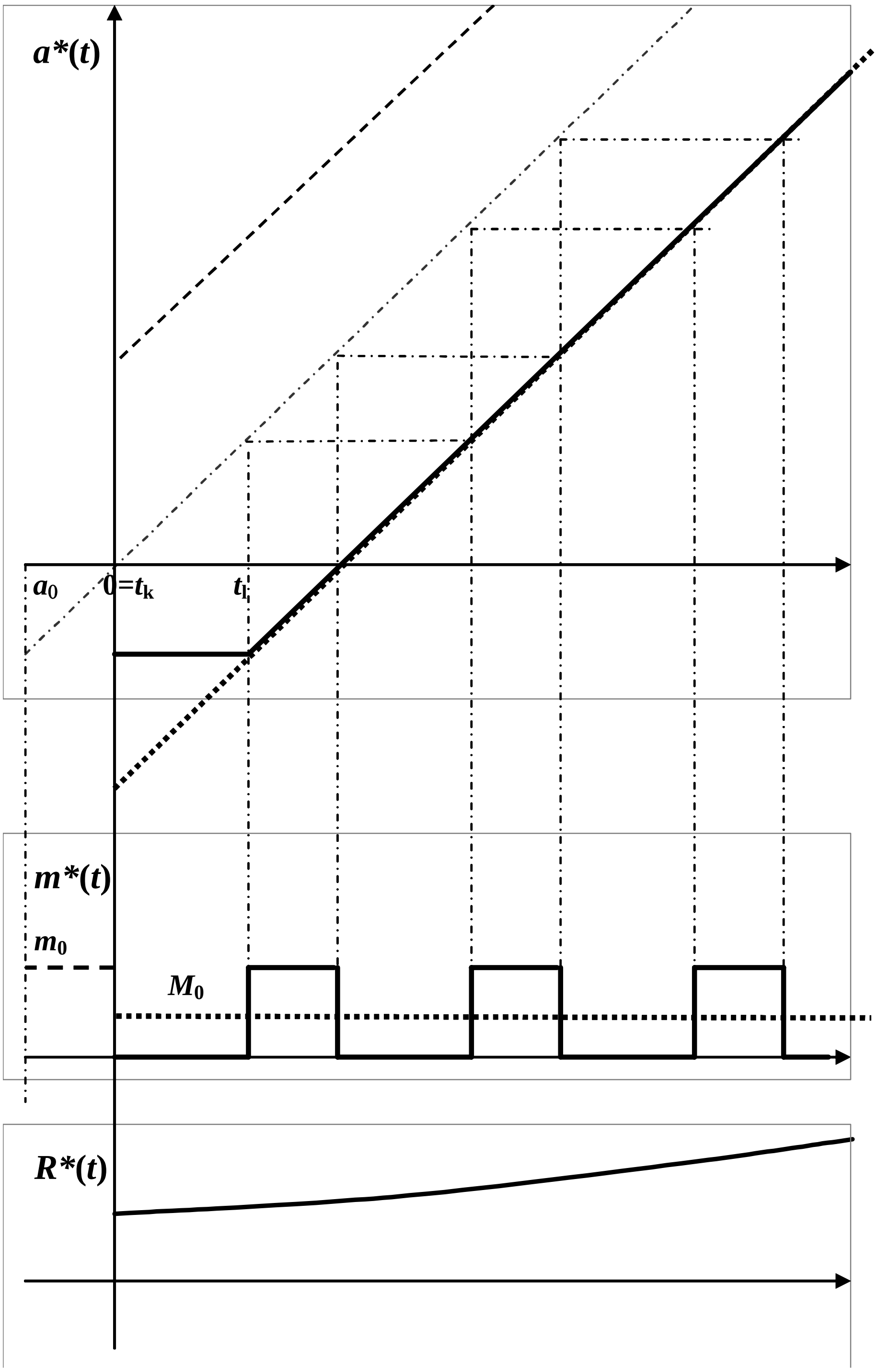

Figure 2. Transition and long-term dynamics under active environment regulation from Example 2 (at specific initial conditions a_0 and m_0). The dotted lines indicate the BGP regime. The dashed line shows the inverse function a^{-1}.

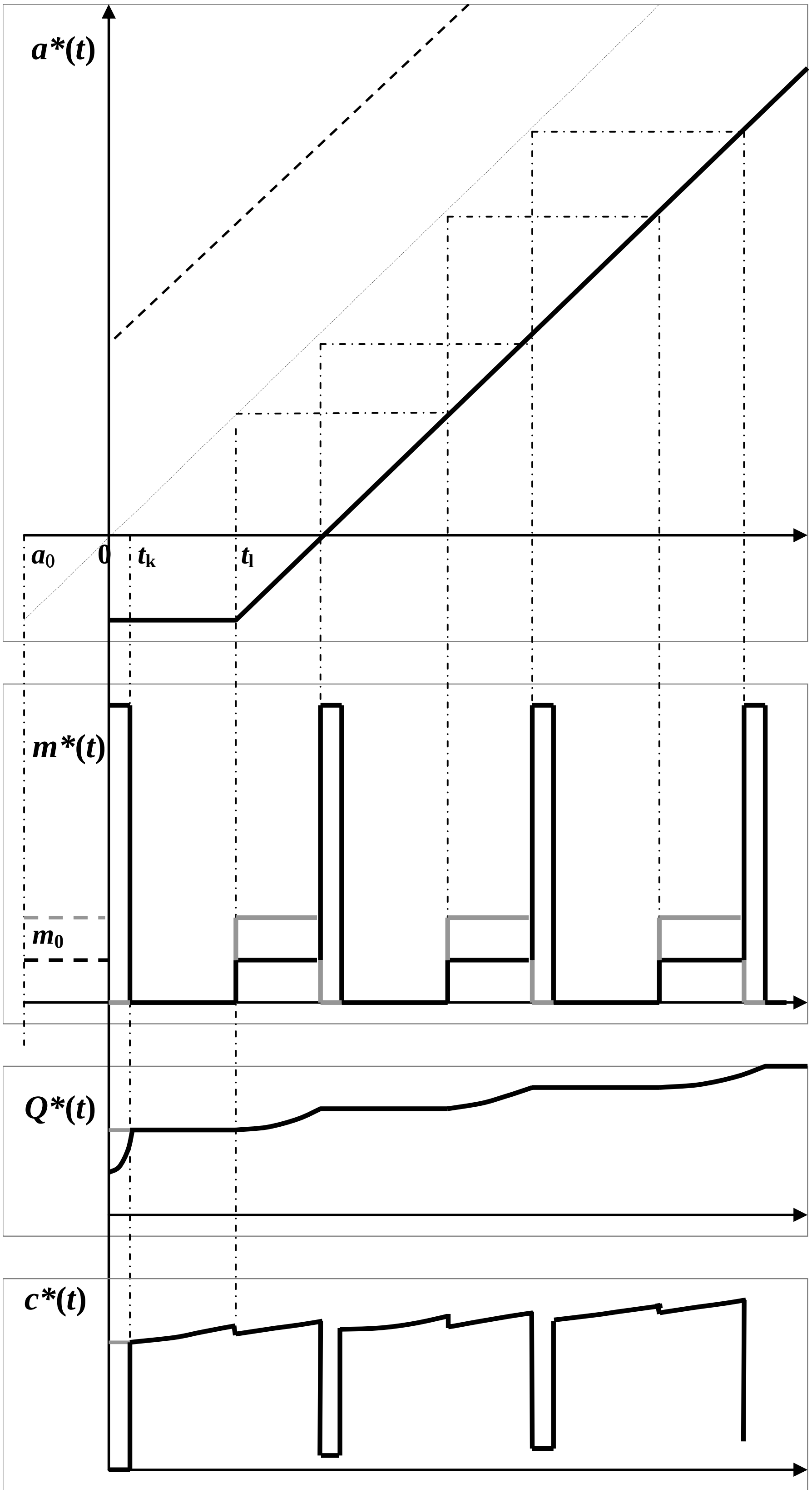

Figure 3. Transition and long-term dynamics under inactive environment regulation from Example 3. The optimal dynamics at active regulation (Example 2) is shown in grey color.