Liquidity Risk, Credit Risk and the Money Multiplier*

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Abstract

Before the financial crisis there was a significant, negative relationship between the money multiplier and the risk free rate; post-crisis it was significant and positive. We develop a model where banks' reserves mitigate not only liquidity risk, but also default/credit risk. When default risk dominates, the model predicts a positive relationship between the risk free rate and the money multiplier. When liquidity risk dominates, that relationship is negative. We suggest reduced liquidity risk, from QE and remunerated reserves, helps explain the multiplier data. The model's implications linking the stock market and the money multiplier are also deduced and verified.

JEL Classification: E40; E44; E50; E51.
Keywords: Liquidity risk; credit risk; excess reserves; US money multiplier, remuneration of reserves

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1. Introduction

The impact on the economy of monetary policy changes depends in part on how broad money, created by the banking system, responds. However, the central bank has direct influence only over narrow money. Consequently, central banks have a keen interest in understanding the determinants of the broad money multiplier. Goodhart (2009) and Williams (2011) are recent, interesting policy-oriented discussions of the multiplier[1].

Understanding the money multiplier is especially important, and challenging, following the recent financial crisis. As happened during The Great Contraction of 1929-33 (Friedman and Schwartz, 1963, Chapter 7), a sharp fall in the broad money multiplier occurred as the recent crisis took hold (see Figure 1 below) and banks contracted lending portfolios and increased reserves.[2] That elevated level of reserves continues eight years after the crisis and provides grounds for suspicion that there may have been a change in banks' behaviour.

However, that level shift in the multiplier has been accompanied by another, less observed, change in the behaviour of the multiplier. Before the financial crisis there was a significant, negative relationship between the money multiplier and the risk free rate. In the post-crisis period that relationship has turned significantly positive suggesting that credit can respond positively to rising short-term interest rates, other things constant. The benchmark model of banks' liquidity risk management cannot easily explain that change in the sign of the relationship.

The conventional approach to modelling banks' liquidity management may be

[1] The importance of the money multiplier has, of course, been analyzed in a huge literature. Recent contributions inlcude: Bernanke and Blinder (1988), Freeman and Kydland (2000) and more recently still Abrams (2011).

[2] See the interesting analysis of von Hagen (2009) comparing the recent trends in monetary aggregates and multipliers amongst the US, UK and the Euro area and with the historical experience of the US during what Friedman and Schwartz (1963) called The Great Contraction.

traced back to the insights of Orr and Mellon (1961).[3] It assumes that banks hold reserves solely for liquidity purposes. The demand for reserve balances then declines in the opportunity cost of holding such reserves, reflected in the loan interest rate, and increases in the penalty rate, modelled below as a premium over the risk free rate. In this framework, an increase in the risk free rate induces a rise in the penalty rate and increases the cost of insufficient reserves holdings. To avoid costly penalties, banks hold more reserves. Therefore, the liquidity risk management model predicts that the broad money multiplier depends negatively on the risk free rate. The model thus has difficulty explaining both the levels shift in the multiplier and the post-crisis positive relationship, documented below, between the risk free rate and the broad money multiplier.

We suggest that the main purpose of holding reserves post-crisis may have switched from liquidity-risk management to credit-risk management.[4] It was the failure of the Fed decisively to support the banking system during the early part of the Great Depression that Friedman and Schwartz (1963) argued set the scene for an unnecessarily deep and prolonged contraction in economic activity.[5] Responding to the recent financial crisis and, many might argue, learning lessons from The Great Contraction, the Fed implemented so-called quantitative easing (QE) measures significantly increasing the excess reserve ratio and reducing the

[3]See also Selgin (2001). Freixas and Rochet (1999) provide a textbook exposition of the model of liquidity risk management and discuss several extensions. They do not explore the link between liquidity risk management and the money multiplier.

[4]Our general model is designed to reflect any joint distribution of liquidity and credit risks. However, we think that recent QE and other policies significantly reduced liquidity risk and we now observe a banking sector that is relatively much more exposed to credit risk; we present below some evidence to support this contention. Arguably, then, the current economic conjuncture provides a unique opportunity to study the mechanism of deposit expansion in an environment with very low liquidity risk.

[5]Of course, the debate continues as to why the Great Depression unfolded in the way it did, but most analysts would accord a major role in the early years of the Depression to the waves of bank failures and the tight liquidity that ensued.

likelihood of, and potential fallout from, additional serious liquidity shocks. In addition, and significantly, excess reserves may also have increased due to the reserves remuneration policy introduced by the Fed after the crisis.

We extend the benchmark model to introduce a role for solvency considerations; banks in the model, as in reality, face uncertainty both because liquidity is difficult to forecast and also because their loan portfolio may perform poorly. Consider the case where banks keep reserves solely to manage credit risk. We assume that banks maximize profits subject to a solvency constraint. That means that the probability of being insolvent must not exceed some limit (perhaps a self-imposed limit, perhaps imposed by regulators). In an economy with a low reserve requirement ratio, the solvency constraint is binding, and the relationship between the risk free investment and the risk free rate is negative. The risk free asset is used to hedge solvency risk. When the return on the safe asset increases, it permits increased investment in risky assets without violating the solvency constraint. That leads to an increase in the money multiplier–the opposite to when liquidity concerns are the only issue.

In short, we show that when liquidity-risk considerations give way to credit-risk management as the dominant driver of reserves accumulation, a positive relationship ensues between the risk free rate and the money multiplier. That in turn implies a positive correlation between the risk free rate and broad money: An increase in the Fed Funds rate can lead to broad money expansion.

We estimate the relationship between changes in the money multiplier and changes in the T-bills rate. It is shown that there is a strong, negative and significant correlation in the pre-crisis period and a strong, positive and robust correlation in the post-crisis period. Our interpretation of that pattern is that liquidity risk was more important for banks before the crisis, but less important than credit risk after the crisis[6]. As suggested, that may reflect continuous QE–like support for liquidity, including remuneration of reserve accounts,[7] and a simultaneous increase in business risk in the post crisis period. The model we develop also turns out to predict that the money multiplier should depend positively on the stock market return and negatively on stock market volatility. We present evidence that the model appears consistent with the data along those dimensions too.

The rest of the paper has the following structure. Section 2 sets out the basic data to motivate the analysis. Section 3 sets out and extends the standard model of banks. Here banks face risks from deposit withdrawals and from solvency concerns. We then focus on two special cases; the first when only liquidity risks are present and the second when only solvency risks are present. The latter version of the model predicts that the money multiplier should depend positively on the risk free rate and the stock market return and negatively on stock market volatility. Section 4 provides a more detailed econometric analysis of the pre- and post-crisis data tests indicating that solvency was less of an issue pre-crisis but a dominating factor post-crisis. The implications of the model for the correlation between the stock market and the money multiplier are also worked out and confirmed against the data. Section 5 presents some evidence on dynamics using VAR estimates. Section 6 concludes.

2. Data: A preliminary pass

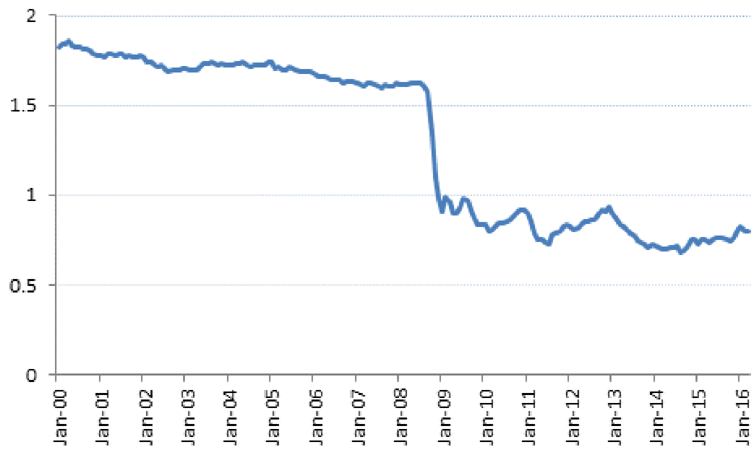
Figure 1 shows the M1 broad money multiplier before and after the its dramatic reduction in 2008. The multiplier fell sharply as the financial crisis unfolded, the

[6] Data presented below indicate evidence of a structural change in the dynamics at the time of the crisis: Checkable deposits did not grow much but were rather volatile before the crisis, but started growing in the post-crisis period.

[7] For an interesting review of QE measures in OECD countries, see Gambacorta et al. (2014).

turning point being September 2008 around the time of the collapse of Lehman Brothers[8]. As von Hagen (2009) notes, a similar pattern was observed in the US as the Great Depression unfolded. It is also worth observing that following its reduction, it appears that the multiplier became somewhat more volatile.

Figure 1. Structural break in the M1 money multiplier



As the target Fed Funds Rate was lowered in effect to zero, one might have expected the money multiplier to recover, at least partially. That did not happen. Moreover, structural shifts aside, the next two charts indicate that the correlation between the T-bill rate and the money multiplier also changed pre- and post-crisis. Figures 2A and 2B look at the periods before and after the collapse in the multiplier. In the earlier period, ignoring possible trends, their appears to be a broadly negative relationship between these two variables. It is most apparent around October 1986, September 1993, and the period of the early 2000's when the pronounced decrease in the T-bills rate was accompanied by a slight increase in the multiplier.

[8]Lehman's filed for Chapter 11 protection on 15 September.

Figure 2B shows what has happened to these two variables since the multiplier has bottomed out, post-September 2008. It suggests that the relationship appears to have changed, becoming positive over much of the sample.

Figure 2A. T-bills rate and M1 money multiplier (right axis)

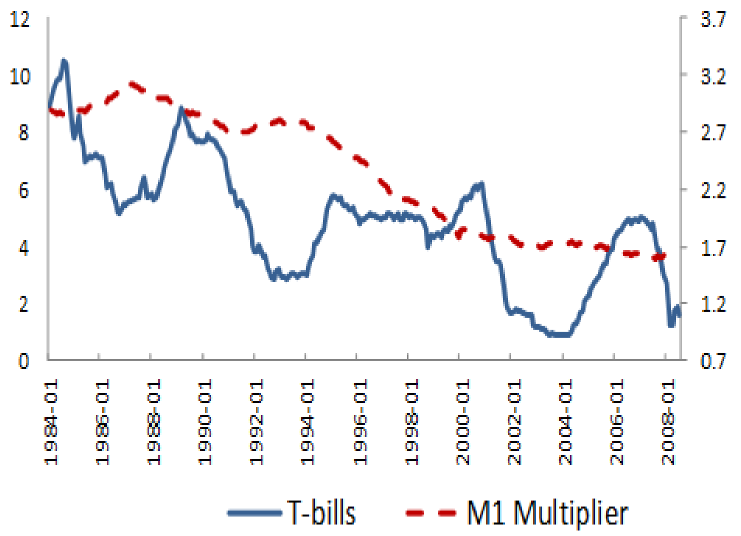
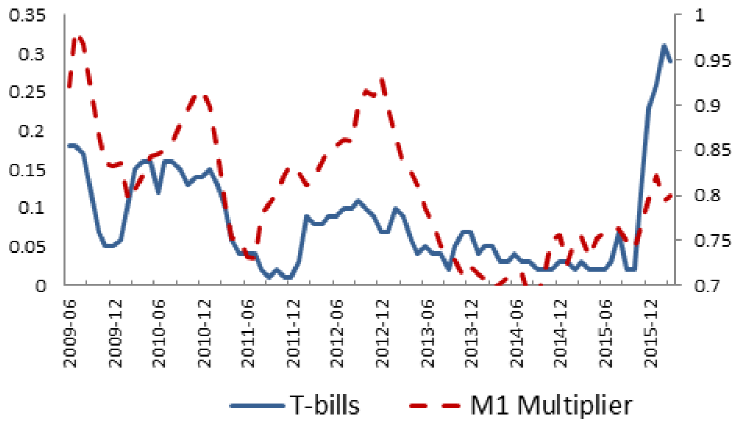


Figure 2B. T-bills rate and M1 money multiplier (right axis)



Of course, these two charts can only be used as a motivation for a more precise quantitative analysis. Consequently, below we present a more careful econometric description of the data that confirms our informal analysis of the data: accounting for heteroskedasticity, there is a significant, negative relationship between the money multiplier and the risk free rate before the financial crisis, whilst post-crisis it is significant and positive.

However, before presenting those results, it is now useful to set out a simple model of bank behavior in order to understand more clearly the link between liquidity risk management and the money multiplier. This shows why the liquidity risk management perspective predicts that the relationship between the risk free rate and the multiplier should be negative, as observed in the earlier sample period above. It also motivates our incorporation into the model of a credit/default-risk objective.

3. Model of a bank facing liquidity and solvency risks

Constructing a model to formalize the bank's problem in the face of liquidity *and* solvency risks takes a little work. A representative bank starts the period with an amount of deposits, D, from households and businesses. The bank may lend to a productive firm who has investment opportunities. It is impossible for the banks (or the firms) to tell ex ante how profitable the firm will be. However, banks do know what the average return will be on a dollar lent. Let B^c denote a bank's risky corporate loans portfolio with average stochastic gross return, R_t^c.

Whatever the banks do not lend to the corporate sector, is kept in the form of reserves: $R = D - B^c$. Part of these reserves are needed to meet liquidity demands such as deposit withdrawal. Let y be the liquidity demand to deposits ratio. That ratio is a stochastic variable with cumulative distribution function G. If reserves are larger than the bank's liquidity needs, the bank can lend the difference on the interbank market and earn the risk free rate $r^f(1-\Delta^L) > 0$ per unit invested, where $\Delta^L \in (0,1)$ is a transaction cost due to lending on the interbank market.[9] Otherwise the bank needs to borrow at rate $r^f\left(\Delta^B+1\right)$, where $\Delta^B > 0$ is a transaction cost associated with borrowing on the interbank market. Thus, at the end of the period a representative bank will earn expected income of $E_t\left\{\max(R-yD,0)r^f(1-\Delta^L)+B^cR^c-\max(yD-R,0)r^f\left(\Delta^B+1\right)\right\}$ and will face costs of D. Here we assume, purely for algebraic simplicity, that no interest is paid on the deposits.

If there were no other constraints facing the bank, its optimal program would simply be

$$\max_{B^c,R,}\Pi = E\left[\max(R-yD,0)r^f(1-\Delta^L)+B^cR^c+R-\max(yD-R,0)r^f\left(\Delta^B+1\right)-D\right]. \tag{3.1}$$

In addition to liquidity shocks we assume that banks manage their balance sheets so that there is also a target probability for solvency. Specifically, banks desire to be solvent with probability $1-\overline{a}$, where $\overline{a}$ is the probability of default. Thus, the bank is assumed to respect the following constraint:

$$\Pr\left\{\left[\max\left(r-y,0\right)r^f(1-\Delta^L)+(1-r)R^c+r-r^f\left(\Delta^B+1\right)\max(y-r,0)-1\right]<0\right\}\leq\overline{a}, \tag{3.2}$$

where we define $r=R/D<1$, the reserve ratio.

That constraint includes two stochastic variables, R^c, the risky return, and y, the liquidity shock. Therefore we need to work with a joint distribution function. Let $g\left(y,R^c\right)$ denote the joint density of y and R^c. We make no specific assumptions as to whether these shocks are correlated or not, and in this sense our set-up is general. One can now transform the solvency constraint into a more convenient mathematical form as formulated in the following proposition.

[9] If it is completely impossible to lend the excess reserves on the interbank market, then $\Delta^L=1$. In the case of reserves remuneration, if the interest rate on reserves equals the risk free rate, then $\Delta^L=0$.

Proposition 3.1. *Solvency constraint (3.2) is equivalent to (3.3):*

$$\int_0^r \left[\int_{-\infty}^{1-A_R(1-\Delta^L)} g\left(y, R^c\right) dR^c \right] dy + \int_r^1 \left[\int_{-\infty}^{1-A_R(r,y)(1+\Delta^B)} g\left(y, R^c\right) dR^c \right] dy = \overline{a}, \qquad (3.3)$$

where $A_R\left(r, y\right) \equiv \frac{(r-y)r^f}{(1-r)}$;

Proof. *In an Appendix we consider two cases; one where* $y < r$ *and another when* $y > r$. *Taken together, these correspond to the first and the second terms of (3.3).* ■

3.1. Liquidity risk management

Consider first the case when solvency is not a problem and constraint (3.2) may be ignored. Then we have a model of pure liquidity risk management and the intermediary's optimization problem is:

$$\max_r \frac{\Pi}{D} = \left(1 - \Delta^L\right) r^f \int_0^r (r-y) dG(y) + (1-r) ER^c + r - r^f \left(1 + \Delta^B\right) \int_r^1 (y-r) dG(y) - 1, \quad (3.4)$$

where $dG(y) = \int_{-\infty}^{+\infty} g\left(y, R^c\right) dR^c$ is the marginal density of y.

The first order condition with respect to the reserve ratio is

$$\begin{aligned} \frac{\partial}{\partial r} \frac{\Pi}{D} &= \left(1 - \Delta^L\right) r^f G(r) - ER^c + 1 + r^f \left(1 + \Delta^B\right) (1 - G(r)) \\ &= r^f \left(1 + \Delta^B\right) - ER^c + 1 - \left(\Delta^B + \Delta^L\right) r^f G(r), \end{aligned} \qquad (3.5)$$

and an internal solution exists if, and only if, condition (3.6) is true:

$$1 > G(r) = 1 - \frac{ER^c - (1 + r^f)}{\left(\Delta^B + \Delta^L\right) r^f} > 0. \qquad (3.6)$$

The left-hand inequality of (3.6) simply implies that the expected return on commercial loans is higher than the risk free rate. If that condition is violated there follows, in effect, a flight to quality where banks do not make commercial loans and instead keep all assets in the form of reserves. The right-hand inequality means that the risk premium for commercial lending should be smaller than the penalty premium $\Delta r^f = \left(\Delta^B + \Delta^L\right) r^f$. If that is not the case, commercial lending is so profitable that reserves are kept at their minimum level.[10]

It is straightforward to see that when condition (3.6) is satisfied, the reserve ratio increases with the risk free rate and penalty Δ. Therefore, when banks are solely concerned with liquidity risk management, the model implies a *negative* relation between the risk free rate and the money multiplier.

However, in the months after the financial crisis there was, arguably, a decline in Δ due to the adoption of so-called Quantitative Easing (QE) and other liquidity-oriented policies, notably remuneration of reserves[11] (Williams, 2011). Also, the Fed Funds rate was reduced almost to zero. Therefore, one should have observed an increase of the money multiplier. However, the money multiplier did not rebound after the crisis as is apparent in Figure 1 above.

The counterfactual prediction of the model in this time period indicates that a sole focus on liquidity management issues is misleading. In other words, the simple view of banking behavior implicit in the traditional liquidity management model, i.e., (3.4), is missing issues that have become important in practice. As Williams (2011) notes:

> Now banks earn interest on their reserves at the Fed .... This fundamental change in the nature of reserves is not yet addressed in our textbook models of money supply and the money multiplier. ...If the interest rate paid on bank reserves is high enough, then banks

[10] Formula (3.6) represents the classic condition for reserves management. In the numerator one sees the expected spread between investment and the reserves holding return, and in the denominator the penalty rate. That formula is identical to formula (8.12) in Freixas and Rochet (2008).

[11] In our model, the policy of reserves remuneration simply reduces Δ^L.

> no longer feel such a pressing need to "put those reserves to work." In fact, banks could be happy to hold those reserves as a risk-free interest-bearing asset, essentially a perfect substitute for holding a Treasury security. If banks are happy to hold excess reserves as an interest-bearing asset, then the marginal money multiplier on those reserves can be close to zero.
>
> In other words, in a world where the Fed pays interest on bank reserves, traditional theories that tell of a mechanical link between reserves, money supply, and ultimately inflation no longer hold. In particular, the world changes if the Fed is willing to pay a high enough interest rate on reserves.

The model developed in the paper may be used to explore that new world and investigate the relationship between the risk free rate and the money multiplier. We argue that the predominant driver of reserves holdings may have shifted from liquidity management to the preservation of solvency. The model can explain some key empirical facts about the money multiplier. It shows that the reserve ratio could actually decline in the level of reserves remuneration. In particular, when reserves are safe assets, the income so generated provides some insurance against risky investments and allows for more lending without compromising solvency. When solvency risk is important, the lending share not only increases with the loan rate but also declines in its risk. Therefore, the money multiplier is affected by stock market behaviour and variations in stock market volatility[12]. That provides another reason why monetary authorities may wish to be alert to stock market developments. We explore these issues further below.

That focus on solvency concerns in banking became more important in the

[12] Merton (1974) explains the positive relationship between stock market volatility and credit risk. See below.

period following the financial crisis. There has been, and will likely continue to be for a few more years, increasing formal capital requirements on banks. Moreover, tighter financial regulations are accompanying a political backlash against financial bailouts. And, at least in some countries, bailouts may be less likely in the near future in part as governments repair public sector balance sheets following the great recession. Such considerations complicate the analysis of the money multiplier further.

We now turn to the bank's problem when the solvency constraint is binding.

3.2. Credit risk management

As mentioned above, QE measures may have significantly reduced the penalty cost, Δ^B, while the reserve remuneration policy cut Δ^L. If Δ is zero then constraint (3.3) specializes to the case where

$$a(r, r^f) = \int_0^1 \left[\int_{-\infty}^{1-\frac{(r-y)r^f}{(1-r)}} g\left(y, R^c\right) dR^c \right] dy = \bar{a}. \tag{3.7}$$

The bank's profit maximization problem then reduces to

$$\max_{r,} E\frac{\Pi}{D} = \left[(r - Ey)\, r^f + (1-r)(ER^c - 1) \right]. \tag{3.8}$$

It can readily be established that constraint (3.7) is binding if $\frac{\partial}{\partial r} E\frac{\Pi}{D} < 0$, which is equivalent to $ER^c > r^f + 1$.

Now define the conditional cumulative probability function of risky returns by $F(y, X) = \int_{-\infty}^{X} g\left(y, R^c\right) dR^c$, which is the marginal cumulative distribution function; its derivative with respect to X is positive $F_x(y, X) = dF(y, X)/dX \geq 0$. Then constraint (3.7) is

$$a(r, r^f) = \int_0^1 F\left[y, 1 - \frac{(r-y)\, r^f}{1-r} \right] dy = \bar{a}. \tag{3.9}$$

Our goal is to investigate the relationship between reserves and the risk free rate. When constraint (3.9) is binding, we can apply the implicit function theorem and compute

$$\frac{dr}{dr^f} = -\frac{\partial a/\partial r^f}{\partial a/\partial r}. \tag{3.10}$$

It is easy to see that

$$\partial a/\partial r = -\int_0^1 F_x\left[y, 1 - \frac{(r-y)\,r^f}{(1-r)}\right]\frac{(1-y)r^f}{(1-r)^2}dy < 0. \tag{3.11}$$

Inequality (3.11) simply reflects that the probability of default declines as bank reserves rise; a higher reserves ratio means a lower probability of default.

As far as the relationship between default and the risk free rate goes, one finds that:

$$\partial a/\partial r^f = \int_0^1 F_x\left[y, 1 - \frac{(r-y)\,r^f}{(1-r)}\right]\frac{(y-r)}{(1-r)}dy. \tag{3.12}$$

It follows then that expression (3.12) is negative if, and only if,

$$\widetilde{E}y < r, \tag{3.13}$$

where $\widetilde{E}y$ is the expected liquidity shock with respect to measure $\widetilde{g}(y) := \frac{F_x\left[y,1-\frac{(r-y)r^f}{(1-r)}\right]}{\int_0^1 F_x\left[y,1-\frac{(r-y)r^f}{(1-r)}\right]dy}$.

Condition (3.13) means that on average banks hold more reserves than the expected liquidity requirement. In reality, the large reserve ratio observed, in combination with QE, leads us to believe that condition (3.13) is likely to be satisfied. In that case, the probability of default declines with the risk free rate: $\partial a/\partial r^f < 0$. Intuitively, then, (3.13) and $\partial a/\partial r^f < 0$ mean that if banks' risk free asset holdings are positive on average, solvency improves with the return on those assets. Consequently, we have proved the following Proposition:

Proposition 3.2. *When the solvency constraint is binding and* $\widetilde{E}y < r$*, banks' reserves ratios decline with the risk free rate.*

Proof. It follows immediately from (3.10) and (3.11). ■

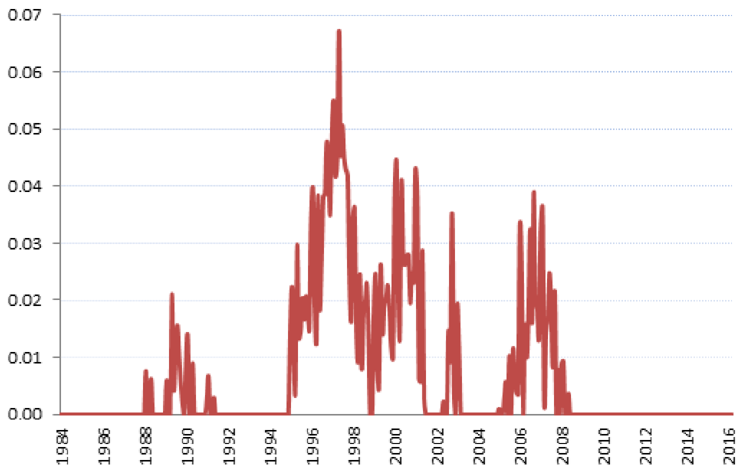
Proposition 3.2 shows that the reserve ratio declines in the risk free rate. The intuition is straightforward: When the risk free investment generates higher returns, it permits a rebalancing of the portfolio towards risky assets without violation of the solvency constraint. And that positive relationship between the risk free rate and the money multiplier has an important policy implication. When liquidity management is less important than credit risk management, increases in the policy interest rate may be consistent with broad monetary expansion.

Condition $\widetilde{E}y < r$ is essential for the positive relation between money multiplier and the risk free rate. That condition became more likely to obtain in the post-crisis period as the distribution of liquidity shocks, proxied by the change in checkable deposits, suggests. Figure 3 shows that for about 12 years before the crisis, the quantity of checkable deposits was stable. Surely related to QE, it started growing after the crisis.

Figure 3. Checkable deposits, billions of dollars

In Figure 4 we plot the percentage decrease in checkable deposits over the previous 12 months conditional on it being positive. It can be seen that deposits have been growing since 2009, consistent with the view that the probability of a liquidity shock has been rather low since the crisis.

Figure 4. Liquidity shock (fall in the deposits)



In the following section we return to look at the empirical relationship between the risk free rate and the money multiplier. We will show, confirming our earlier and rather informal look at the data that, after the crisis, the money multiplier moves positively and significantly with the risk free rate.

4. Money multiplier and the risk free rate

The model of pure credit risk management discussed in the previous section predicts that the money multiplier positively depends on the risk free rate. That is because safe assets may be used for hedging against risky lending. Ceteris paribus, when the risk free rate is higher, the cash flow from the safe asset is larger and more risky lending is possible without violation of the solvency constraint. That simple model predicts that the money multiplier ought to increase in the risk free interest rate. Visual inspection of the data post 2009 in Table 2B above appears to indicate that there is indeed some positive co-movement in the money multiplier and the risk free rate in the US.

To back up that casual empiricism, we also estimated various econometric (variance function) models which permit a wide range of persistent processes to characterize the variance. Table 1 shows that the M1 multiplier ($MM1_t$) follows a very persistent, heteroskedastic process and indeed the movement of the money multiplier is positively related to the movement of the 3 month T-Bill rate (TB_t).

Table 1. Money Multiplier and T-Bill rate post-crisis period: Sep. 2008– May 2017

Dependent variable $MM1_t$

Independent variables	Model 1	Model 2	Model 3	Model 4	Model 5
Intercept	0.141^{****}	0.087^{****}	0.119^{****}	0.097^{****}	0.099^{****}
$MM1_{t-1}$	0.808^{****}	0.885^{****}	0.843^{****}	1.176^{****}	1.205^{****}
TB_t	0.075^{****}	0.040^{****}	0.048^{****}	0.048^{****}	0.154^{****}
$MM1_{t-2}$				-0.304^{****}	-0.332^{****}
TB_{t-1}					-0.125^{****}
Variance Equations					
	EGARCH (1,0)	GARCH (0,1)	IGARCH (1,1)	EGARCH (1,1)	EGARCH (1,1)
C_1	-8.15^{****}	$8.82E^{-05****}$	0.20^{****}	-0.40^{****}	-0.42^{****}
C_2	1.13^{****}	0.797^{****}	0.80^{****}	-0.28^{****}	-0.32^{****}
C_3				0.92^{****}	0.91^{****}
R^2	0.927	0.911	0.922	0.937	0.947
Akaike	−4.29	−4.475	−4.30	−4.67	−4.664
Schwartz	−4.16	−4.349	−4.20	−4.49	−4.463
Durbin-Watson	1.69	1.07	1.16	1.69	2.02
Log likelihood	230.2	239.9	229.85	252.2	252.9

Data Source: St Louis Fed. **** indicates significance at the 1% level, *** indicates significance at the 5% level, ** indicates significance at the 10% level

As noted, all versions of the models in Table 1 imply that the multiplier is highly autocorrelated and that it is significantly related to the T-Bill rate. The variance equations also indicate significant heteroskedasticity although the information criterion statistics do not provide a strong guide as between the models. We run different specifications to check for the robustness of the sign with different lags of TB_t and found that it is indeed robust to model specification.

Before performing the regressions reported in Table 1 we checked the M1 multiplier data and rejected the unit root hypothesis for the post crisis period. Unsurprisingly, however, we cannot reject the unit root hypothesis for the complete sample from 1984 onwards. Consequently, we respecified the model in differences. So we define $dMM1_t$ as the first difference in the money multiplier and dTB_t as the first difference in the T-Bill rate. Table 2 shows that the relationship between the increase in the money multiplier and the increase in the risk free rate is once again positive and highly significant. Moreover, it remains robust to the choice of the model for the residual process.

Table 2. Change in M1 money multiplier, post-crisis

Dependent variable $dMM1_t$

Independent variables	Model 6	Model 7	Model 8	Model 9	Model 10
$dMM1_{t-1}$	0.415^{****}	0.273^{****}	0.30^{****}	0.20^{****}	0.24^{****}
dTB_t	0.208^{****}	0.311^{****}	0.08^{***}	0.13^{****}	0.12^{****}
dTB_{t-1}			0.10^{**}	0.19^{****}	0.19^{****}
Variance Equations					
	GARCH (1,0)	EGARCH (1,1)	GARCH (0,1)	EGARCH (1,0)	IGARCH (1,1)
C_1	${3.8E^{-04}}^{****}$	-10.94^{****}	${6.4.8E^{-05}}^{***}$	-7.83^{****}	0.12^{****}
C_2	0.43^{***}	1.029^{****}	0.84^{****}	0.52^{***}	0.88^{****}
C_3		-0.37^{****}			
R^2	0.545	0.458	0.575	0.657	0.653
Akaike	−4.489	−4.431	−4.587	−4.489	−4.517
Schwartz	−4.338	−4.305	−4.461	−4.363	−4.416
Durbin-Watson	2.223	1.913	1.883	1.944	2.013
Log likelihood	239.7	237.67	245.8	240.7	241.19

4.1. Money multiplier and the stock market

As indicated earlier, the theoretical model developed here has implications for the relationship between the money multiplier and stock market behaviour. Following Merton's (1974) model for valuing credit risks, corporate debt is equivalent to a risk free investment less a European call option on common stock. Since banks can diversify their assets we assume that their loan portfolios can be priced against the stock market index.

Consider (3.7) under the assumption that R^c is lognormally distributed, $R^c \sim LN\ (r^c, \sigma^2)$ and independent of y. The proportion of non-performing loans will be defined as

$$a(r, r^f) = \int_0^1 g_y(y) \left(\Pr \left[R^c < 1 - \frac{r-y}{1-r} r^f \right] \right) dy = \overline{a}. \tag{4.1}$$

That expression may be rewritten as

$$a(r, r^f, r^c, \sigma) = \int_0^1 g_y(y) \Phi \left[\frac{\ln \left(1 - \frac{r-y}{1-r} r^f\right) - r^c}{\sigma} \right] dy = \overline{a}, \tag{4.2}$$

where Φ is the normal CDF. Moreover

$$\frac{\partial a(r, r^f, \mu, \sigma)}{\partial r^c} = -\frac{1}{\sigma} \int_0^1 g_y(y) \Phi' \left[\frac{\ln \left(1 - \frac{r-y}{1-r} r^f\right) - r^c}{\sigma} \right] dy < 0, \tag{4.3}$$

is always negative and in combination with (3.11) allows us to apply the implicit function theorem to conclude that $\frac{dr}{r^c} < 0$. It follows immediately that the money multiplier increases in stock returns. Similarly, if reserves are sufficiently greater than liquidity needs then $\frac{\partial a(r, r^f, \mu, \sigma)}{\partial \sigma} > 0$[13]. Therefore, our model predicts both

[13] The necessary and sufficient condition is:

$$\frac{\partial z(r, r^f, \mu, \sigma)}{\partial \sigma} = -\frac{1}{\sigma^2} \int_0^1 g_y(y) \Phi' \left[\frac{\ln \left(1 + \frac{(y-r) r^f}{(1-r)}\right) - \mu}{\sigma} \right] \left(\ln \left(1 - \frac{(r-y) r^f}{(1-r)}\right) - \mu \right) dy > 0.$$

that the money multiplier increases in stock market returns and that it declines in stock market volatility. The intuition for these predictions is straightforward: As returns rise solvency concerns present less of a constraint increasing lending. On the other hand, increasing volatility of returns presents more of a risk to solvency.

We computed the market return (RET_t) using the S&P500 index and used the VIX data as a proxy for volatility (VIX_t). The regression results in Table 3 are estimated in first differences. Overall, the data support the predictions of the model during the post-crisis time period.[14].

[14] We also found that the effective Fed Funds rate (FED_t) performs slightly better than the Treasury Bills rate in explaining post-crisis behaviour of the money multiplier. When we replace the T-Bill rate with FED_t, the R^2 statistic and the log-likelihood became larger, whilst the Schwartz and Akaike criteria decline slightly.

Table 3. M1 multiplier and the stock market-post crisis period

Dependent variable $dMM1_t$

Independent variables	Model 11	Model 12	Model 13	Model 14	Model 15
$dMM1_{t-1}$	0.4597^{****}	0.4474^{****}	0.4419^{****}	0.4512^{****}	0.3369^{****}
$dVIX_{t-1}$	-0.0016^{***}			-0.0010^{****}	-0.0007^{****}
$dRET_{t-2}$		0.0640^{**}	0.0545^{****}	0.0828^{****}	0.0427^{****}
dTB_t	0.1102^{****}	0.1013^{****}	0.1386^{****}	0.092^{****}	0.1020^{****}
$dMM1_{t-2}$					-0.0671^{***}
dTB_{t-1}					0.0971^{****}
Variance Equations					
	EGARCH (1,1)	EGARCH (1,1)	EGARCH (1,2)	EGARCH (1,1)	EGARCH (1,1)
C_1	-11.20^{****}	-11.13^{****}	-8.21^{****}	-12.6^{****}	-13.1^{****}
C_2	0.936^{****}	1.172^{****}	1.639^{****}	1.198^{****}	1.51^{****}
C_3	-0.41^{***}	-0.38^{****}	-0.24^{****}	-0.54^{****}	-0.56^{****}
C_4			0.31^{****}		
R^2	0.549	0.482	0.518	0.515	0.618
Akaike	−4.478	−4.463	−4.6412	−4.749	−4.702
Schwartz	−4.326	−4.312	−4.4612	−4.570	−4.475
Durbin-Watson	2.216	2.035	2.16	2.115	2.065
Log likelihood	241.11	240.34	250.66	245.50	255.90

4.2. Pre-crisis estimation

We also estimated the relationship between the money multiplier, the stock market and the risk free rate in the pre-crisis period beginning in January 1990, when data for the VIX became available, up until July 2008. We find that there is a significant negative trend in the M1 multiplier before the crisis. But more importantly, there is a negative, significant and robust relation between the money multiplier and the risk free rate (Table 4). Viewed through the lens of our model, it appears that in the pre-crisis period credit risk and the solvency constraint were less important than liquidity risk in influencing aggregate bank behaviour. That may in part reflect a presumption of bailouts and lax regulation on the part of banks. In any event, there is a significant positive relation between the money multiplier and the market return and a significant negative relation with the volatility of the stock market.

Figure 1 indicates that before the crisis the M1 multiplier declined over time. That is often attributed to development of financial instruments with a positive return which has similar liquidity properties as M1 but is classified as M2. Therefore we add a constant to account for that trend. We find that it is indeed negative and highly significant. We also find that the asymmetric model for variance explains better the volatility of the money multiplier compared with the symmetric specification[15]. We found that over this (pre-crisis) sample the lagged T-Bills rate has more explanatory power compared with the contemporaneous rate. It is also the case that the VIX is very important in explaining the money multiplier, but that the stock return is much less significant.

[15] We employed the asymmetric EGARCH (0,1,1) model with the ARCH parameter constrained to zero. The variance equation is: $\log(\sigma_t^2) = C_1 + C_3 u_{t-1}\sigma_{t-1} + C_4 \log(\sigma_{t-1}^2)$

Table 4. Money multiplier in pre-crisis period. Sample: March 1990-July 2008.

Dependent variable $dMM1_t$

Independent variables	Model 16	Model 17
$const$	-0.0048^{****}	-0.0042^{****}
$dMM1_{t-1}$	0.1527^{***}	0.1920^{****}
$dVIX_{t-1}$	-0.0008^{****}	-0.0008^{****}
dTB_{t-1}	-0.0133^{****}	-0.0120^{****}
Variance Equations		
	EGARCH (1,1)	T-EGARCH (0,1,1)
C_1	-3.698^{****}	-1.602^{****}
C_2	0.279^{****}	-1.129^{****}
C_3	0.607^{****}	0.820^{****}
R^2	0.097	0.0957
Akaike	-5.965	-5.986
Schwartz	-5.857	-5.879
Durbin-Watson	1.946	2.031
Log likelihood	663.21	665.52

5. Evidence from VARs

Finally, we look at some evidence from VAR estimation as a way to characterize the causality and dynamics between the risk free rate and money multiplier. Tables 5 and 6 report results for the following 4-variable VAR: ($dMM1_t$, $dRET_t$, dTB_t, $dVIX_t$). From the VAR, we again find a strong negative relationship from the risk free rate to the money multiplier in the pre-crisis period. And in the post-crisis estimates, we again see the key difference. There is a clear positive relationship running from the T-Bills rate to the money multiplier (Figure 5, right hand panel).

Figure 5. Response of $dMM1_t$ to dTB_{t-1}

Pre-crisis Post-crisis

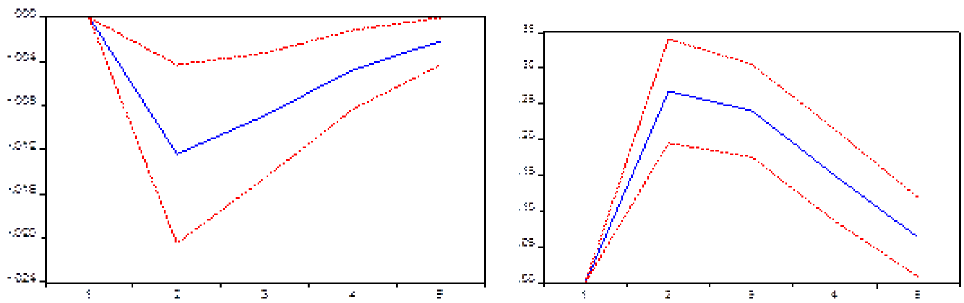


Table 5. Pre-crisis VAR estimation: Period: March 1990-July 2008

	$dMM1_t$		dTB_t		$dRET_t$		$dVIX_t$	
$dMM1_{t-1}$	0.18^{****}	[2.80]	0.135	[0.13]	0.23	[0.95]	-24.64*	[-1.67]
dTB_{t-1}	-0.012^{****}	[-3.06]	0.447^{****}	[7.22]	-0.006	[-0.42]	1.51*	[1.66]
$dRET_{t-1}$	0.001	[0.06]	-0.19	[-0.826]	-0.43^{****}	[-7.63]	-13.7^{****}	[-4.01]
$dVIX_{t-1}$	-0.0007^{****}	[-2.34]	-0.003	[-0.823]	0.006^{****}	[5.49]	-0.018	[-0.26]
$Const$	-0.005^{****}	[-5.38]	-0.01	[-0.99]	0.001	[0.28]	-0.10	[-0.47]
R^2	0.10		0.21		0.35		0.10	

Table 6. Post-crisis VAR estimation: Period: October 2008-May 2017

	$dMM1_t$		dTB_t		$dRET_t$		$dVIX_t$	
$dMM1_{t-1}$	0.14***	[1.80]	-0.23**	[-1.70]	-0.20	[-1.32]	52.55^{****}	[4.47]
dTB_{t-1}	0.26^{****}	[7.34]	0.567^{****}	[8.87]	-0.006	[-0.42]	-25.99^{****}	[-4.70]
$dRET_{t-1}$	0.003	[-0.06]	-0.035	[-0.826]	-0.43^{****}	[-7.63]	-22.29^{****}	[-3.05]
$dVIX_{t-1}$	-0.0020^{****}	[-3.09]	-0.005^{****}	[-4.958]	0.003^{****}	[2.645]	0.0026	[0.027]
R^2	0.64		0.67		0.21		0.29	

5.1. Money, liquidity and uncertainty

Whilst interpreting the VAR results with our model is difficult in part as the latter is static and the former dynamic, some progress may be made. Reinterpret the solvency condition as a constraint that should be satisfied *on average.* In this case, the expected default probability α_{t-1} will positively impact future risk as measured by the VIX_t. More specifically, consider again our formula for solvency

$$\begin{aligned}\alpha\left(r, r^f\right) &= \int_0^r \left[\int_{-\infty}^{1-\frac{(r-y)r^f}{(1-r)}(1-\Delta^L)} g\left(y, R^c\right) dR^c\right] dy \\ &+ \int_r^1 \left[\int_{-\infty}^{1-\frac{(r-y)r^f}{(1-r)}\left(1+\Delta^B\right)} g\left(y, R^c\right) dR^c\right] dy. \end{aligned} \tag{5.1}$$

Even if the distribution of risky returns (R^c) and liquidity shocks (y), are stochastically independent, formula (5.1) connects the default risk, α, to liquidity risk.

Proposition 5.1. *If the liquidity shock and the risky return are independent,* $g\left(y, R^c\right) = g^y\left(y\right) \times g^{R^c}\left(R^c\right)$, *where* $g^y\left(y\right)$ *and* $g^{R^c}\left(R^c\right)$ *are* $pdfs$ *for* y *and* R^c, *the following is true:*
i) First order stochastic dominance in R^c *reduces the probability of default; and*
ii) first order stochastic dominance in y *increases the probability of default.*

Proof. Define the CDF for the risky return as $G^{R^c}(x) = \int\limits_{-\infty}^{x} g^{R^c}(R^c)dR^c$, then formula (3.3) becomes

$$\alpha\left(r, r^f\right) = \int\limits_0^r g^y\left(y\right) G^{R^c}\left[1 + \frac{\left(y-r\right)\left(1-\Delta^L\right)r^f}{\left(1-r\right)}\right] dy + \int\limits_r^1 g^y\left(y\right) G^{R^c}\left[1 + \frac{\left(y-r\right)\left(1+\Delta^B\right)r^f}{\left(1-r\right)}\right] dy \tag{5.2}$$

Consider two distributions for R^c. If $G_1^{R^c}(x) < G_2^{R^c}(x)$ for all x,then $G_1^{R^c}$ stochastically dominates $G_2^{R^c}$, and from (5.2) $G_1^{R^c}$ implies a smaller probability of default. That completes the proof of the first statement.

Expression (5.2) can also be written as

$$\alpha\left(r, r^f\right) = \int\limits_0^r g^y\left(y\right) \widetilde{G}^{R^c}\left[y\right] dy \tag{5.3}$$

where

$$\widetilde{G}^{R^c}\left[y\right] = \begin{cases} G^{R^c}\left[1 + \frac{(y-r)(1-\Delta^L)r^f}{(1-r)}\right] & \text{if } y \leq r \\ G^{R^c}\left[1 + \frac{(y-r)(1+\Delta^B)r^f}{(1-r)}\right] & \text{if } y > r \end{cases}$$

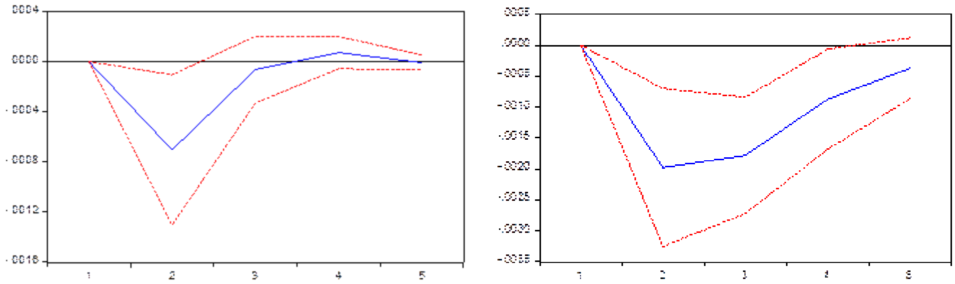
which is a strictly increasing function. Therefore the second statement follows from the property of first order stochastic dominance. ■

Proposition 5.1 shows that even if investment return is independent of liquidity shocks, credit risk, measured by the default probability α, increases in liquidity risk. Indeed, that effect would be even more pronounced if we were to assume, as documented in Acharya and Pedersen (2005), that risky returns are negatively correlated with investor sentiments concerning liquidity risk. In our VAR estimation we do not have a proxy for the liquidity shock, however it is reasonable to expect it to be reflected in the volatility index, VIX; that is, an increase in liquidity risk should result in a higher VIX and a lower broad money multiplier. That is probably one of the channels that explains why one observes a lower money multiplier after an increase in the VIX both before and after the crisis periods. Tables 5 and 6 also show that uncertainty reduces lending either because of liquidity risk or default risk. However, after the crisis, the quantitative response is almost 2.5 times larger and more persistent (Figure 6).

Figure 6. Response of $dMM1_t$ to $dVIX_{t-1}$

Pre-crisis period

Post-crisis



The negative relation between volatility and expected return is well documented in the empirical literature starting with the seminal paper of Campbell and Hentschel (1992). Our VAR estimation is consistent with that observation: increases in VIX_{t-1} lead higher future market return RET_t in both samples. Finally, as noted by Campbell and Hentschel (1992), "volatility is typically higher after the stock market falls than after it rises, so stock returns are negatively correlated with future volatility." Again, that appears consistent with our VAR estimation as an increase in RET_{t-1} leads a decline in the VIX_t, with the coefficients negative and highly significant for both samples. Our VAR estimates appear consistent with known stylized facts from financial econometrics: Figures 7 and 8 show the response of $dRET_t$ to $dVIX_{t-1}$ (it is positive and consistent with the intuition that "risk requires compensation") and the response of $dVIX_t$ to $dRET_{t-1}$ (it is negative, suggesting that "good news reduces uncertainty").

Figure 7. Response of $dRET_t$ to $dVIX_{t-1}$

Pre-crisis | Post-crisis

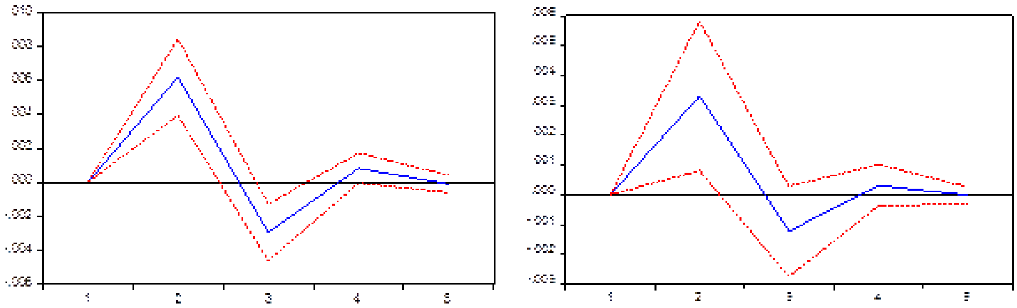
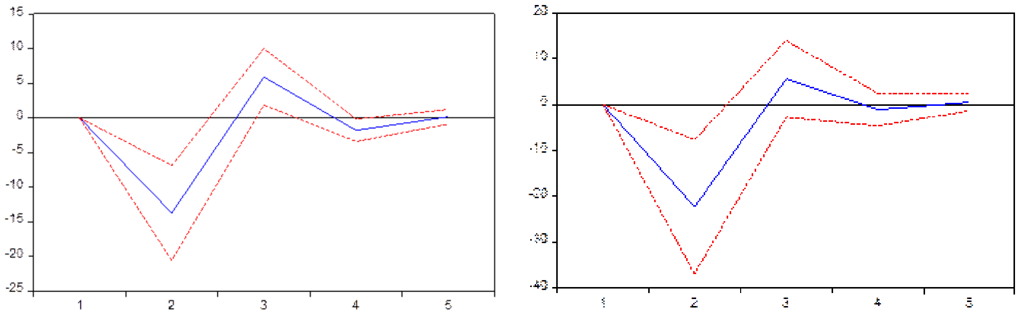


Figure 8. Response of $dVIX_t$ to $dRET_{t-1}$

Pre-crisis | Post-crisis



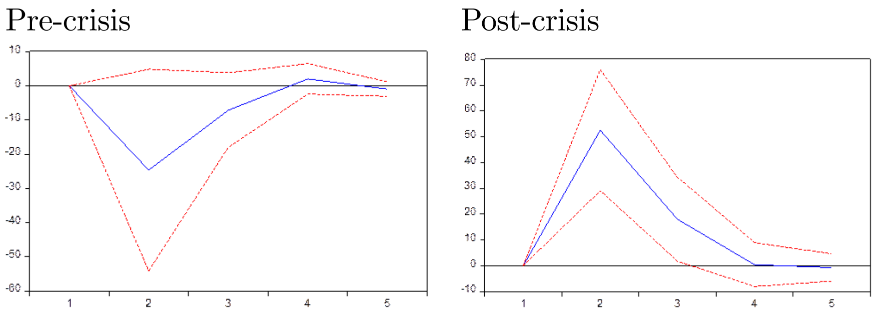
Consequently, it is interesting to investigate the relationship between the reserve ratio and default predicted by our model. It follows that

$$\begin{aligned} \frac{d\alpha\left(r, r^f\right)}{dr} &= -\frac{(1-\Delta^L)r^f}{(1-r)^2}\int_0^r \left[g\left(y, 1-A_R(1-\Delta^L)\right)\right](1-y)\,dy \\ &\quad -\frac{(1+\Delta^B)r^f}{(1-r)^2}\int_r^1 \left[g\left(y, 1-A_R\left(r,y\right)\left(1+\Delta^B\right)\right)(1-y)\right]dy. \end{aligned}$$

When the risk free rate is positive, this expression is strictly negative. Therefore one would expect to observe the VIX_t increase after an increase in the money multiplier, $MM1_{t-1}$. One does indeed observe such a relationship after the crisis in Table 6 (Figure 9, right panel).

The pre-crisis relationship between these variables is less significant. However, the overall explanatory power of the regression is low. The lack of significance of $dVIX_t$ to $dMM1_{t-1}$ reported in Table 5 (Figure 9, left) may be due to the inability of the market to anticipate the possibility of banking default in the pre-crisis period. In other words, the market did not take into full consideration the stability of the banking sector and that is why the reserve ratio did not appear directly to affect the VIX before the crisis.

Figure 9. Response of $dVIX_t$ to $dMM1_{t-1}$



The relationship between default risk and the risk free rate is less obvious. Direct differentiation implies:

$$\frac{d\alpha\left(r, r^f\right)}{dr^f} = -\frac{(1-\Delta^L)r^f}{(1-r)}\int_0^r \left[g\left(y, 1 - A_R(1-\Delta^L)\right)\right](r-y)\,dy \tag{5.4}$$

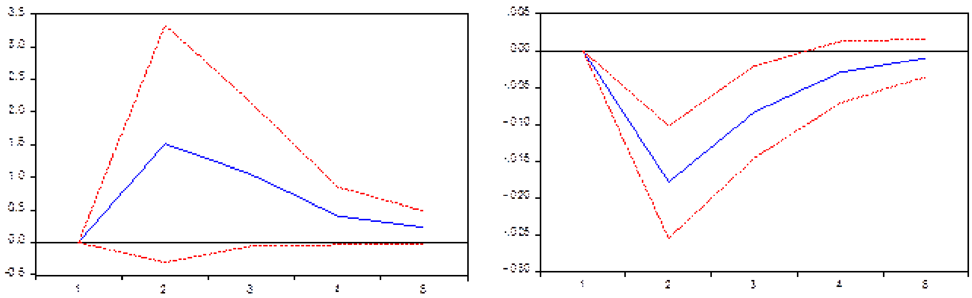
$$+\frac{(1+\Delta^B)r^f}{(1-r)}\int_r^1 \left[g\left(y, 1 - A_R\left(r, y\right)\left(1+\Delta^B\right)\right)(y-r)\right]dy. \tag{5.5}$$

The sign of this expression is ambiguous. The first line is negative and the second line is positive. The negative part (5.4) represents an "insurance effect"; that is, where reserves are larger than the liquidity shock. In this case, a larger risk free rate implies higher profit and therefore a reduced risk of default. The second line

(5.5) represents a "penalty effect"; that is, when reserves are insufficient, $(y > r)$. Now, a higher risk free rate implies a larger penalty for a given liquidity shortage; the end result is that it increases costs and reduces net profit leading to a higher probability of default. We suggests that the second effect, the "penalty effect", may have been relatively more significant before the crisis when the reserve ratio was sufficiently low and this is why one observes a positive, although insignificant, relationship from the risk free rate to the VIX. After the crisis, as the reserve ratio increased, the first effect, the "insurance effect", became more important and we observe a large, negative and significant effect from the risk free rate to the VIX (see Figure 10).[16]

Figure 10. Response of $dVIX_t$ to dTB_{t-1}

Pre-crisis Post-crisis



6. Summary and conclusions

The behaviour of the US broad money multiplier has changed pre- and post-financial crisis. In particular, post crisis there was a significant, positive relationship between the money multiplier and the risk free rate. A benchmark model of banks' liquidity risk management cannot readily explain that change in the sign of the relationship post-crisis. Consequently, we develop a model of bank

[16]Table 7 shows that effect is robust to adding the stock market return and that the overall explanatory power of the regression rises substantially to that addition.

behaviour that includes a solvency or credit risk management objective for banks, alongside a liquidity risk management objective. Such an extension turns out to be able to match the patterns we observe in the data. The model we developed also had additional empirical implications for how the stock market affects the multiplier. These implications appear also to match the data.

We make two final observations. First, as noted, the model, emphasizing the primacy of credit risk, predicts that the money multiplier increases with the risk free rate. That is in stark contrast to the typical view based on the traditional model of liquidity risk management which our model nests. We modelled that result empirically and found evidence of a positive relation between the money multiplier and the T-bills rate in the post crisis period. We also found that the relation was negative in the period 1990-2008. We conjecture that after the crisis, liquidity constraints were significantly relaxed by various QE programmes. At the same time, credit risk became a more significant issue for banks in part as regulatory increases in capital requirements, and other measures, were introduced. That may explain the apparent change in the aggregate of banks' lending strategies, and therefore in the behaviour of the money multiplier. An implication of these findings is that an increase in the target Fed Funds rate, in combination with QE, may lead to an increase in the money multiplier and may not be as contractionary as in the pre-financial crisis world.

Second, the model also predicts some systematic relationships between the money market and the stock market. Specifically, we found that the money multiplier is positively related to stock returns and negatively to stock volatility. Therefore stock market movements may be an important consideration for monetary policymakers in assessing monetary conditions in the wider economy. We think our analysis suggests further study of these linkages are warranted.

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7. Appendix

Proof of Proposition (3.1)

We consider 2 cases.

The first case is when the liquidity shock is smaller than reserves, $y < r$. In that case violation of solvency (3.2) will occur if $(r-y)\, r^f (1-\Delta^L) + (1-r) R^c + r - 1 < 0$ which may be rewritten as

$$R^c < 1 - r^f (1-\Delta^L) \frac{r-y}{1-r}. \tag{7.1}$$

The 'partial' constraint (7.1) gives us some insight on the relationship between the risk free rate and reserves. If reserves are sufficient to meet liquidity demand, $r > y$, then an increase in r^f would allow the bank to reduce the reserve ratio, r, keeping the right hand side constant.

For any realization of the liquidity shock, y, one may compute the conditional probability of default

$$\Pr\left[R^c < 1 - \frac{(r-y)\, r^f (1-\Delta^L)}{1-r}\right] = \int_{-\infty}^{1-A_R(1-\Delta^L)} g(y, R^c)\, dR^c,$$

where $A_R(r,y) \equiv \frac{(r-y) r^f}{(1-r)}$. And so the probability of default given that the liquidity shock is smaller than reserves is

$$\Pr\left[R^c < 1 - \frac{(r-y)\, r^f (1-\Delta^L)}{1-r};\ \text{and } y < r\right] = \int_0^r \left[\int_{-\infty}^{1-A_R(1-\Delta^L)} g(y, R^c)\, dR^c\right] dy. \tag{7.2}$$

The second case is when the liquidity shock is larger than reserves, $y > r$. In that case a similar manipulation shows that the probability of default when the liquidity shock exceeds reserves is

$$\Pr\left[R^c < 1 + \frac{r^f \left(1+\Delta^B\right)(y-r)}{1-r};\ \text{and } y > r\right] = \int_r^1 \left[\int_{-\infty}^{1-A_R(r,y)\left(1+\Delta^B\right)} g(y, R^c)\, dR^c\right] dy. \tag{7.3}$$