Optimal Time-Consistent Monetary, Fiscal and Debt Maturity Policy*

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Abstract

We develop a New Keynesian model with government bonds of mixed maturity and solve for optimal time-consistent policy using global solution techniques. This reveals several non-linearities absent from LQ analyses with one-period debt. Firstly, the steady-state balances an inflation and debt stabilization bias to generate a small negative debt value with a slight undershooting of the inflation target. This falls far short of first-best ('war chest') asset levels. Secondly, starting from debt levels consistent with currently observed debt to GDP ratios the optimal policy will gradually reduce that debt, but the policy mix changes radically along the transition path. At high debt levels there is a reliance on a relaxation of monetary policy to reduce debt through an expanded tax base and reduced debt service costs, while tax rates are used to moderate the increases in inflation. However, as debt levels fall, the use of monetary policy in this way diminishes and the authority turns to fiscal policy to continue debt reduction. This endogenous switch in the policy mix occurs at higher debt levels, the longer the average debt maturity. Allowing the policymaker to optimally vary debt maturity in response to shocks and across varying levels of debt, we find that variations in maturity are largely used to support changes in the underlying time-consistent policy mix rather than the speed of fiscal correction. Finally, introducing a mild degree of policy maker myopia can reproduce steady-state debt to GDP ratios and inflation rates not dissimilar to those observed empirically, without changing any of the qualitative results presented in the paper.

Key words: New Keynesian Model; Government Debt; Monetary Policy; Fiscal Policy; Credibility; Time Consistency; Maturity Structure.

JEL codes: E62, E63

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1 Introduction

The recent global financial crisis has led to an unprecedented peacetime increase in government debt in advanced economies. Figure 1 shows that the debt to GDP ratios in advanced economies steadily increased from 73% in 2007 to 105.3% in 2014. This development has prompted an interest from both policy makers and researchers in rethinking the appropriate relationship among monetary policy, fiscal policy and debt management policy. The conventional policy assignment calls upon monetary authorities to determine the level of short-term interest rates in order to control demand and inflation, while the fiscal authorities choose the level of the budget deficit to ensure fiscal sustainability and a debt management office undertakes the technical issue of choosing the maturity and form in which federal debt is issued. With the onset of the 2007/2008 financial crisis and the subsequent easing of monetary policy, the clean lines between these domains have blurred. With short-term interest rates at the zero lower bound, central banks have resorted to quantitative easing (QE) to support aggregate demand. Because QE shortens the maturity structure of debt instruments that private investors have to hold, central banks have effectively entered the domain of debt-management policy.[1] At the same time, fiscal authorities' debt-management offices have been extending the average maturity of the debt to mitigate fiscal risks associated with the government's growing debt burden. These fiscal actions have operated as a kind of reverse quantitative easing, replacing money-like short-term debt with longer-term debt.[2] The observation that monetary and fiscal policies with regard to government debt have been pushing in opposite directions suggests the need to reconsider the principles underlying the optimal combination of monetary, fiscal and debt management policies.

Against this background, this paper studies jointly optimal monetary and fiscal policy when the policy makers can issue a portfolio of bonds of multiple maturities, but cannot commit. A major focus of the paper is on how the level and maturity of debt affects the optimal time-consistent policy mix and equilibrium outcomes in the presence of distortionary taxes and sticky prices. From this analysis, we can draw some conclusions on questions like whether surprise inflation and interest rates are likely to be used, in addition to adjustments to taxes and government spending, in order to reduce and stabilize debt. Given the magnitude of the required fiscal consolidation in so many advanced economies, how the policy mix is likely to change as debt is stabilized from these levels is highly relevant.

In sticky price New Keynesian models with one-period government debt, Schmitt-Grohe and Uribe (2004b) show that even a mild degree of price stickiness implies nearly constant inflation and near random walk behavior in government debt and tax rates when policy makers can commit to time-inconsistent monetary and fiscal policies, in response to shocks. In other words, monetary policy should not be used to stabilize debt. However, Sims (2013) questions the robustness of this result when government can issue long-term nominal bonds. With only short-term government debt, unexpected current inflation or deflation is the only way to change its market value in cushioning fiscal shocks. In contrast, if debt is long term, large changes in the value of debt can be achieved through

[1] The series of open market operations by the Federal Reserve between 2008 and 2014 and the expansion in excess reserves reduced the average duration of U.S. government liabilities by over 20%, from 4.6 years to 3.6 years (Corhay et al., 2014)

[2] For instance, the stock of US government debt with a maturity over 5 years that is held by the public (excluding the holdings by the Federal Reserve) has risen from 8 percent of GDP at the end of 2007 to 15 percent at the middle of 2014 (Greenwood et al., 2014).

sustained movements in the nominal interest rate, with much smaller changes in current inflation. Based on these considerations, Sims sketches out a theoretical argument for using nominal debt - of which the real value can be altered with surprise changes in inflation and interest rates - as a cushion against fiscal disturbances to substitute for large movements in distorting taxes. This mechanism is explored further in Leeper and Leith (2017).

Our paper contributes to the literature along at least three dimensions. Firstly, we take both non-state-contingent short-term and long-term nominal bonds into account. The consideration of long-term debt and the maturity structure is motivated by Sims' theoretical insights as well as the empirical facts. Figure 2 (right panel) shows the average debt maturity in a selection of advanced countries is between 2 and 14 years. Moreover, in section 5.2.4, we extend our analysis to encompass a situation where the policy maker can optimally vary the maturity structure as part of the policy problem.

Secondly, we focus on the time-consistent policy problem which is less studied in the literature, and allow for the possibility that the policy maker may suffer some degree of 'myopia' as a means of capturing the frictions in fiscal policy making highlighted by the political economy literature (see Alesina and Passalacqua (2017) for a recent survey of the political economy of government debt). In contrast, Sims' arguments for using surprise inflation as a complement to tax adjustments were made in the context of an environment where the policy-maker could commit. In a linear-quadratic approximation to the policy problem, Leith and Wren-Lewis (2013) show that the time inconsistency inherent in commitment policy means that the optimal time-consistent discretionary policy for debt is quite different. The random walk result, typically, no longer holds, and instead debt returns to its steady-state level following shocks. In addition, time-consistent policy regime is arguably the more appropriate description of policymaking around the world. While the Ramsey policy implies it is optimal to induce a random walk in steady state debt as a result of the standard tax smoothing argument, ex ante fiscal authorities typically want to adopt fiscal rules which are actually quite aggressive in stabilizing debt. They then typically abandon these rules in the face of adverse shocks (see, for example, Calmfors and Wren-Lewis (2011) for details on the numerous breaches of the Stability and Growth Pact in Europe even prior to the financial crisis). There is, therefore, a clear failure to adopt fiscal rules which mimic commitment policy. Understanding how optimal time-consistent (possibly myopic) discretionary policy differs from its time-inconsistent counterpart, in particular in its implications for debt dynamics, has particular empirical relevance today as governments assess the extent to which they need to reverse the large increases in debt caused by the severe recession, in a context where fiscal policy commitments are often far from credible.

Thirdly, we solve the model non-linearly using the global solution methods which enable us to analyze episodes with sharp increases in debt to GDP ratios as observed in many countries during the global recession. Leith and Wren-Lewis (2013) adopt traditional linear-quadratic (LQ) methods by using an artificial device to ensure the steady-state is efficient and then linearizing the model around this steady state, while Schmitt-Grohe and Uribe (2004b) adopt a second-order approximation to the first-order conditions of the Ramsey problem.[3] In contrast, we are not imposing any kind of approximation around

[3]In LQ models with long-term debt, Leeper and Zhou (2013) ask some similar questions and they solve the optimal monetary and fiscal policies from the timeless perspective, while Bhattarai et al. (2014) study optimal time-consistent monetary and fiscal policies, taking the zero lower bound constraint on nominal interest rate into consideration.

a steady-state so that we can fully explore the effects of non-linearities inherent in the New Keynesian model.[4] In fact, the results under discretion in Leith and Wren-Lewis (2013) suggest that there are massive non-linearities - for example, there is an overshooting in the debt correction in a single period when the economy is linearized around higher steady state debt levels, but a more gradual debt reduction following shocks when the linearization takes place around a lower debt to GDP ratio. This implies that the speed of debt stabilization is likely to be highly state dependent.

To address these non-linearities and the time-inconsistency problem which depends on the incentives to induce inflation surprises to stimulate the economy and/or deflate debt, we develop a New Keynesian model augmented with fiscal policy and a portfolio of mixed maturity bonds and solve the optimal time-consistent policy problem using global non-linear solution techniques. In particular, we study whether and how nominal government debt maturity affects optimal monetary and fiscal policy decisions and equilibrium outcomes in the presence of distortionary taxes and sticky prices. In the model, the government cannot commit, and would like to use unexpected inflation in two ways. Firstly, as implied by the usual inflation bias problem, the policy maker faces a temptation to boost inflation in order to stimulate an economy where production is sub-optimally low due to monopolistic competition and tax distortions. Since tax rates are endogenous in the model, the extent of the inflationary bias problem is endogenous as well. Secondly, the policy maker faces a second bias in that surprise inflation can affect the real value of the outstanding stock of nominal government liabilities. In this way, the government faces the additional temptation to increase inflation in order to reduce its debt burden. Anticipating this, economic agents raise their inflationary expectations - following Leith and Wren-Lewis (2013) - we label this the 'debt stabilization bias'. This bias is also state contingent in that the efficacy of surprise inflation in stabilizing debt, and therefore the temptation to resort to such a policy, is rising in the level of debt.

We find the following key results. Firstly, the steady-state balances the inflation and debt stabilization biases to generate a small negative long-run optimal value for debt, which implies a slight undershooting of the inflation target in steady state. This falls far short of the accumulated level of assets that would be needed to finance government consumption and eliminate tax and other distortions (the so-called 'war chest' level).

Secondly, starting from levels of debt consistent with currently observed debt to GDP ratios, the optimal time-consistent policy will gradually reduce that debt, but with large increases in inflation and radical changes in the policy mix along the transition path. At high debt levels, there is a reliance on a relaxation of monetary policy to reduce debt through an expansion in the tax base and reduced debt service costs, while tax rates are used to moderate the increases in inflation. However, as debt levels fall, the use of monetary policy in this way is diminished and the policy maker turns to fiscal policy to continue the reduction in debt. This is akin to a switch from an active to passive fiscal policy in rule based descriptions of policy, which occurs endogenously under the optimal policy as debt levels fall. It can also be accompanied by a switch from passive to active monetary policy. This switch in the policy mix occurs at higher debt levels, the longer the average maturity of government debt. The increase in inflation associated with the inflation and debt stabilization biases reduces bond prices. This means that for

[4] There are some recent papers using global solution techniques which consider optimal discretionary monetary policy with trivial fiscal policy in the New Keynesian models, see Anderson et al. (2010), Van Zandweghe and Wolman (2011), Nakata (2013), Leith and Liu (2014), Ngo (2014) and among others.

a given deficit the policy maker will need to issue more bonds, but when debt is of longer maturity, the policy maker will also pay less to repay the existing debt stock. Therefore, with longer maturity debt the desire to reduce debt rapidly along the transition path is reduced and the debt stabilization bias is mitigated. As a result, for a longer debt maturity, ceteris paribus, the policy maker is freer to raise taxes to stabilize debt, as the marginal inflationary costs of such tax increases are lower.

Thirdly, we also consider how the time-consistent policy maker would vary debt maturity, in response to shocks and across varying levels of debt. We show that variations in the maturity structure are optimally used to support alterations in the time-consistent policy mix, rather than support a significantly different speed of fiscal correction.

Finally, we allow for the possibility that the policy maker may by 'myopic' in the sense that they discount the future more heavily than the infinitely lived household, as a means of capturing the short-termism in policy making that may be induced by the political process. We find that this can easily shift steady-state equilibrium debt levels and deviations of inflation from target, from negative to positive in line with observed values, without qualitatively affecting the rest of our conclusions.

Related Literature: Our paper is related to several strands of the optimal monetary and fiscal policy literature. We will discuss those that are most closely related in terms of topics and numerical methods.

Our contribution is most closely related to the literature that studies optimal fiscal and monetary policy in sticky price New Keynesian models using non-linear solution techniques. Following the work of Schmitt-Grohe and Uribe (2004b) and Siu (2004), Faraglia et al. (2013) solve a Ramsey problem using a parameterized expectation algorithm (PEA) to examine the implications for optimal inflation of changes in the level and maturity of government debt. We study the discretionary equivalent of this policy, which is radically different. Niemann and Pichler (2011) globally solve for optimal fiscal and monetary policies under both commitment and discretion in an economy exposed to large adverse shocks. Using the same projection method, Niemann et al. (2013) study time-consistent policy in the model of Schmitt-Grohe and Uribe (2004b) and identify a simple mechanism that generates inflation persistence. Government spending is exogenous in the latter two papers which also do not consider long-term debt. Similarly, abstracting from long-term debt, Matveev (2014) compares the efficacy of discretionary government spending and labor income taxes for the purpose of fiscal stimulus at the liquidity trap. The value function iteration (VFI) method is adopted to deal with the zero lower bound problem. In contrast to these papers, debt of different maturities, time-consistent optimal policy making and endogenous government expenditure are all essential elements in our model.

Aside from the relatively small literature exploring optimal monetary and fiscal policy in non-linear New Keynesian models, there is a vast literature on Ramsey fiscal and monetary policy in the tradition of Lucas and Stokey (1983), which tends to focus on real or flexible-price economies. In flexible-price environments, the government's problem consists in financing an exogenous stream of public spending by choosing the least disruptive combination of inflation and distortionary income taxes. In an incomplete-markets version of Lucas and Stokey (1983), Aiyagari et al. (2002) simulate the model globally and show that the level of welfare in Ramsey economies with and without real state-contingent debt is virtually the same. In addition, they reaffirm the random-walk results of debt and taxes from Barro (1979). Angeletos et al. (2013) introduce collateral constraints and a liquidity role for government bonds into Aiyagari et al. (2002). They use the VFI method to globally solve the modified model and find that the steady-state level of debt is no longer indeterminate, when government bonds can serve as collateral. Cao (2014) extends Angeletos et al. (2013) with long-term debt and studies how the cost of inflation for commercial banks affects the design of fiscal and monetary policy. Likewise, Faraglia et al. (2014) use the PEA methods to solve a Ramsey problem with incomplete markets and long-term bonds. They show that many features of optimal policy are sensitive to the introduction of long-term bonds, in particular tax variability and the long run behavior of debt. Our findings convey the same message that maturity lengths like those observed in actual economies can substantially alter the nature of optimal policies, but the policy problem in our sticky price economy where the policy maker is unable to commit is fundamentally different.

There is also a literature on optimal fiscal and monetary policy in monetary models, which do not contain nominal interia, but may contain a cost to inflation. Schmitt-Grohe and Uribe (2004a) study Ramsey policy in a flexible-price model with cash-in-advance constraint, which essentially extends the model of Lucas and Stokey (1983) to an imperfectly competitive environment. A global numerical method is used to characterize the dynamic properties of the Ramsey allocation. In a cash-in advance model, Martin (2009) studies the time consistency problems that arise from the interaction between debt and monetary policy, since inflation reduces the real value of nominal liabilities. He uses the projection methods to deal with the generalized Euler equations, see also Martin (2011), Martin (2013) and Martin (2014) where time consistent policies are studied in variants of the monetary search model of Lagos and Wright (2005). In contrast, we abstract from monetary frictions and emphasize nominal price stickiness which is the conventional approach to generating sizable real effects from monetary policy.

Moving away from models which jointly model monetary and fiscal policy, there is also a literature on optimal time-consistent fiscal policy in real models. This literature typically focuses on Markov-perfect policy, where households' and the government's policy rules are functions of payoff-relevant variables only. Local approximations around a non-stochastic steady state are typically infeasible for these models, since optimal behavior is characterized by generalized Euler equations that involve the derivatives of some equilibrium decision rules, and thus it is impossible to compute the steady state independent of these rules. Hence, as in our contribution, global solution methods are required. Klein and Rios-Rull (2003) compare the stochastic properties of optimal fiscal policy without commitment with those properties under a full-commitment policy in a neoclassical growth model with a balanced government budget, see also Krusell et al. (2006) and Klein et al. (2008). Ortigueira (2006) studies Markov-perfect optimal taxation under a balanced-budget rule, while Ortigueira et al. (2012) deal with the case of unbalanced budgets. In a version of Lucas and Stokey (1983) model with endogenous government expenditure, Debortoli and Nunes (2013) find that when governments cannot commit, debt is no longer indeterminate and often converges to a steady-state with no debt accumulation at all. This is a quite striking difference in the behavior of debt between the full commitment and the no-commitment cases. Similarly, Grechyna (2013) also considers endogenous government spending in the environment of Lucas and Stokey (1983) with only one-period debt and shows that around the steady state, the properties of the fiscal variables are very similar, regardless of commitment assumptions. More recently, Debortoli et al. (2015) consider a Lucas and Stokey (1983) economy without state-contingent bonds and commitment, and show that the government actively manages its debt positions and can approximate optimal policy by confining its debt instruments to consols.

Our paper shares the same technical problem due to the presence of generalized Euler equations, but nominal rigidities make our model setup quite different from these papers.

Finally, the new political economy literature (see Alesina and Passalacqua (2017) for a comprehensive survey) considers how various aspects of the political process affect the accumulation of government debt, and the tendency of some economies to be prone to a deficit bias. While there are numerous mechanisms through which political economy considerations influence fiscal policy, including the use of debt as a strategic variable, wars of attrition over who bears the burden of fiscal reforms and the nature of the budgetary process itself, in essence these political frictions imply that policy makers may not fully internalize the long-term benefits of lower debt, while remaining acutely aware of the short-term costs of any fiscal correction. We shall capture this implicit myopia informally, by considering the implications of the policy maker discounting the future at a rate which is higher than that of society as a whole.

Roadmap. The paper proceeds as follows. We describe the benchmark model in section 2. The first best allocations are characterized in section 3. We study the optimal time consistent policy problem in section 4. In section 5, we describe the solution method and present the numerical results. In section 6 we conclude.

2 The Model

Our model is a standard New Keynesian model, but augmented to include the government's budget constraint where government spending is financed by distortionary taxation and/or borrowing. This basic set-up is similar to that in Benigno and Woodford (2003) and Schmitt-Grohe and Uribe (2004b), but with some differences. Firstly, we allow the government to optimally vary government spending in the face of shocks, rather than simply treating government spending as an exogenous flow which must be financed. This is a necessary modification to consider issues like assessing the relative efficacy of government spending cuts and tax increases in debt stabilization.[5] Secondly, our nominal debt is not of single-period maturity, but consists of a portfolio of bonds of mixed maturities. In reality, most countries issue long-term nominal debt in overwhelming proportions of total debt. This is an important consideration in highly indebted economies, since even modest surprise changes in inflation and interest rates can have substantial effects on the market value of debt, and hence become a meaningful source of fiscal revenue.[6] This fact suggests that the maturity structure of debt is an essential element in characterizing jointly optimal monetary and fiscal policy. Thirdly, we not only take the average debt maturity as exogenously given, but also allow it to optimally vary over the business cycle, see section C.3 in the appendix. Finally, we capture informally the implications of adding political frictions to the policy making process by assuming the policy maker's time horizon may be shortened as a result of the electoral cycle.

[5] International Monetary Fund (2012) reports that current fiscal consolidation efforts rely heavily on government spending cuts. In addition, Bi et al. (2013) introduce ex ante uncertainty over the composition of the fiscal consolidation, either tax based or spending based, and show that the macroeconomic consequences of spending cuts can be quite different from tax increases, even if the direct fiscal consequences are similar.

[6] See Hall and Sargent (2011) and Sims (2013) for the empirical findings on the contribution of this kind of fiscal financing to the decline in the U.S. debt-GDP ratio from 1945 to 1974.

2.1 Households

There are a continuum of households of size one. Households appreciate private consumption as well as the provision of public goods and dislike labor. We shall assume complete asset markets, such that, through risk sharing, they will face the same budget constraint. As a result the typical household will seek to maximize the following objective function

$$E_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, N_t, G_t\right) \tag{1}$$

where C, G and N are a consumption aggregate, a public goods aggregate, and labour supply respectively.

The consumption aggregate is defined as

$$C_t = \left(\int_0^1 C_t(j)^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}} \tag{2}$$

where j denotes the good's type or variety and $\epsilon > 1$ is the elasticity of substitution between varieties. The public goods aggregate takes the same form

$$G_t = \left(\int_0^1 G_t(j)^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}} \tag{3}$$

The budget constraint at time t is given by

$$\int_0^1 P_t(j)C_t(j)dj + P_t^S B_t^S + P_t^M B_t^M \leq \Xi_t + (1 + \rho P_t^M)B_{t-1}^M + B_{t-1}^S + W_t N_t(1 - \tau_t)$$

where $P_t(j)$ is the price of variety j , Ξ is the representative household's share of profits in the imperfectly competitive firms, W are wages, and τ is an wage income tax rate.[7] Households hold two basic forms of government bond. The first is the familiar one period debt, B_t^S which has the price equal to the inverse of the gross nominal interest rate, $P_t^S = R_t^{-1}$. The second type of bond, following Woodford (2001), is actually a portfolio of many bonds which pay a declining coupon of ρ^j dollars $j+1$ periods after they were issued, where $0 < \rho \leq \beta^{-1}$. The duration of the bond is given by $(1-\beta\rho)^{-1}$, which allows us to vary ρ as a means of changing the implicit maturity structure of government debt. By using such a simple structure, we need only price a single bond, since any existing bond issued j periods ago is worth ρ^j new bonds. In the special case where $\rho = 1$, these bonds become infinitely lived consols, and when $\rho = 0$, the bonds reduce to the familiar single period bonds typically studied in the literature.

Households must first decide how to allocate a given level of expenditure across the various goods that are available. They do so by adjusting the share of a particular good in their consumption bundle to exploit any relative price differences - this minimizes the costs of consumption. Optimization of expenditure for any individual good implies the

[7]Since fiscal policy is one important element of this paper, we do not assume any kind of lump-sum-tax-financed subsidy to offset the distortion arising from monopolistic competition, which is a typical assumption in the optimal fiscal and monetary policy literature using New Keynesian models. Thus, the steady-state of the model economy need not be efficient. In addition, in the presence of the zero lower bound constraint, policy functions have kinks, therefore an accurate evaluation of optimal policy and welfare requires a global solution method.

demand function given below,

$$C_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} C_t$$

where we have price indices given by

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj\right)^{\frac{1}{1-\epsilon}}$$

The budget constraint, therefore, can be rewritten as

$$P_t^S B_t^S + P_t^M B_t^M \leq \Xi_t + (1 + \rho P_t^M) B_{t-1}^M + B_{t-1}^S + W_t N_t (1 - \tau_t) - P_t C_t \tag{4}$$

where $\int_0^1 P_t(j) C_t(j) dj = P_t C_t$. P_t is the current price level. The constraint says that total financial wealth in period t can be worth no more than the value of financial wealth brought into the period, plus nonfinancial income during the period net of taxes and the value of consumption spending.

For much of the analysis, the one period government bond B_t^S is assumed to be in zero net supply with beginning-of-period price P_t^S, while the general portfolio of government bond B_t^M is in non-zero net supply with beginning-of-period price P_t^M. Higher ρ raises the maturity of the bond portfolio. We cannot allow the rate of decay on bonds to become time varying, without either implicitly allowing the government to renege on existing bond contracts or tracking the distribution of bonds of different maturities that have been issued in the past. Therefore, in order to allow the policy maker to tractably vary the maturity structure, we shall in section 5.2.4 consider the case where both B_t^S and B_t^M are potentially in non-zero net supply, so that the policy maker can vary the overall maturity of the outstanding debt stock by varying the relative proportion of short and longer-term bonds in that portfolio.

Similarly, the allocation of government spending across goods is determined by minimizing total costs, $\int_0^1 P_t(j) G_t(j) dj$. Given the form of the basket of public goods, this implies,

$$G_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} G_t$$

2.1.1 Households' Intertemporal Consumption Problem

The first part of the households intertemporal problems involves allocating consumption expenditure across time. For tractability, assume that (1) takes the specific form

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} + \chi \frac{G_t^{1-\sigma_g}}{1-\sigma_g} - \frac{N_t^{1+\varphi}}{1+\varphi}\right) \tag{5}$$

We can then maximize utility subject to the budget constraint (4) to obtain the optimal allocation of consumption across time, based on the pricing of one period bonds,

$$\beta R_t E_t \left\{ \left(\frac{C_t}{C_{t+1}}\right)^{\sigma} \left(\frac{P_t}{P_{t+1}}\right) \right\} = 1 \tag{6}$$

and the declining payoff consols,

$$\beta E_t \left\{ \left(\frac{C_t}{C_{t+1}} \right)^\sigma \left(\frac{P_t}{P_{t+1}} \right) \left(1 + \rho P_{t+1}^M\right) \right\} = P_t^M \tag{7}$$

Notice that when these reduce to single period bonds, $\rho = 0$, the price of these bonds will be given by $P_t^M = R_t^{-1}$. However, outside of this special case, the longer term bonds introduce the term structure of interest rates to the model. It is convenient to define the stochastic discount factor (for nominal payoffs) for later use,

$$\beta \left(\frac{C_t}{C_{t+1}} \right)^\sigma \left(\frac{P_t}{P_{t+1}} \right) = Q_{t,t+1}$$

The second FOC relates to their labour supply decision and is given by,

$$(1 - \tau_t) \left(\frac{W_t}{P_t} \right) = N_t^\varphi C_t^\sigma$$

That is, the marginal rate of substitution between consumption and leisure equals the after-tax wage rate. Besides these FOCs, necessary and sufficient conditions for household optimization also require the households' budget constraints to bind with equality. Defining household wealth brought into period t as,

$$D_t = (1 + \rho P_t^M) B_{t-1}^M + B_{t-1}^S$$

the no-Ponzi-game condition can be written as,

$$\lim_{T \to \infty} E_t \left[\frac{1}{R_{t,T}} \frac{D_T}{P_T} \right] \geq 0 \tag{8}$$

where

$$R_{t,T} = \prod_{s=t}^{T-1} \left(\frac{1 + \rho P_{s+1}^M}{P_s^M} \frac{P_s}{P_{s+1}} \right)$$

for $T \geq 1$ and $R_{t,t} = 1$, also see Eusepi and Preston (2011). The no-Ponzi-game says that the present discounted value of household's real wealth at infinity is non-negative, that is, there is no overaccumulation of debt. In equilibrium, the condition holds with equality.

2.2 Firms

The production function is linear, so for firm j

$$Y_t(j) = A_t N_t(j) \tag{9}$$

where $a_t = \ln(A_t)$ is AR(1) such that $a_t = \rho_a a_{t-1} + e_{at}$, with $0 \leq \rho_a < 1$ and $e_{at} \overset{i.i.d}{\sim} N(0, \sigma_a^2)$. The real marginal costs of production is defined as $mc_t = W_t / (P_t A_t)$. The demand curve they face is given by,

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t$$

where $Y_t = \left[\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj\right]^{\frac{\epsilon}{\epsilon-1}}$. Firms are also subject to quadratic adjustment costs in changing prices, as in Rotemberg (1982).

We define the Rotemberg price adjustment costs for a monopolistic firm j as,

$$v_t(j) = \frac{\phi}{2}\left(\frac{P_t(j)}{\Pi^* P_{t-1}(j)} - 1\right)^2 Y_t \tag{10}$$

where $\phi \geq 0$ measures the degree of nominal price rigidity. The adjustment cost, which accounts for the negative effects of price changes on the customer–firm relationship, increases in magnitude with the size of the price change and with the overall scale of economic activity Y_t.

The problem facing firm j is to maximize the discounted value of profits,

$$\max_{P_t(j)} E_t \sum_{z=0}^{\infty} Q_{t,t+z} \Xi_{t+z}(j)$$

where profits are defined as,

$$\Xi_t(j) = P_t(j)Y_t(j) - mc_t Y_t(j) P_t - \frac{\phi}{2}\left(\frac{P_t(j)}{\Pi^* P_{t-1}(j)} - 1\right)^2 P_t Y_t$$

So that, in a symmetric equilibrium where $P_t(j) = P_t$ the first order conditions are given by,

$$\begin{aligned} 0 &= (1-\epsilon) + \epsilon mc_t - \phi \frac{\Pi_t}{\Pi^*}\left(\frac{\Pi_t}{\Pi^*} - 1\right) \\ &+ \phi \beta E_t \left[\left(\frac{C_t}{C_{t+1}}\right)^{\sigma} \frac{Y_{t+1}}{Y_t} \frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*} - 1\right)\right] \end{aligned} \tag{11}$$

which is the Rotemberg version of the non-linear Phillips curve relationship.

2.2.1 Market Clearing

Goods market clearing requires, for each good j,

$$Y_t(j) = C_t(j) + G_t(j) + v_t(j)$$

which allows us to write,

$$Y_t = C_t + G_t + v_t$$

with $v_t = \int_0^1 v_t(j)\, dj$. In a symmetrical equilibrium,

$$Y_t \left[1 - \frac{\phi}{2}\left(\frac{\Pi_t}{\Pi^*} - 1\right)^2\right] = C_t + G_t$$

There is also market clearing in the bonds market where we assume, initially, that the one period bonds are in zero net supply, $B_t^S = 0$ and the remaining longer term portfolio evolves according to the government's budget constraint which we will now describe.

2.3 Government Budget Constraint

The government consists of two authorities. First, there is a monetary authority which controls the nominal interest rates on short-term nominal bonds. Second, there is a fiscal authority deciding on the level of government expenditures, labor income taxes and on debt policy. Government expenditures consist of spending for the provision of public goods and for interest payments on outstanding debt. The level of public goods provision is a choice variable of the government. We assume that monetary and fiscal policy is coordinated by a benevolent policymaker who seeks to maximize household welfare, and the government can credibly commit to repay its debt. We shall consider the implications of policy maker myopia motivated by political frictions in section 5.2.5.

Government expenditures G_t are financed by levying labor income taxes at the rate τ_t, and by issuing one-period, risk free (non-contingent), nominal obligations B_t^S, and long term bonds B_t^M. The government's sequential budget constraint is then given by

$$P_t^M B_t^M + P_t^S B_t^S + \tau_t W_t N_t = P_t G_t + B_{t-1}^S + (1 + \rho P_t^M) B_{t-1}^M$$

Assuming that the one-period bond is assumed in zero net supply,[8] that is, $B_t^S = 0$, and rewriting in real terms

$$P_t^M b_t = (1 + \rho P_t^M) \frac{b_{t-1}}{\Pi_t} - \frac{W_t}{P_t} N_t \tau_t + G_t \tag{12}$$

where real debt is defined as, $b_t \equiv B_t^M / P_t$.

Given the nominal nature of debt, monetary policy decisions affect the government budget through three channels: first, the nominal interest rate policy of the monetary authority influences directly the nominal return the government has to offer on its instruments; second, nominal interest rate decisions also affect the price level and thereby the real value of outstanding government deb; and third, in our sticky-price economy the real effects of monetary policy can affect the size of the tax base.

In particular, the role of the maturity of government debt can be seen clearly from the government budget constraint. In (12), the amount of outstanding real government debt is $P_t^M b_t$, and the period real return on holding government debt is $(1 + \rho P_t^M) / \left(\Pi_t P_{t-1}^M \right)$. If $\rho = 0$, government debt b_t is reduced to one-period debt, and the only way to adjust the real return on bonds ex post is through inflation in the current period Π_t. Large fluctuations in prices can be very costly in the presence of nominal rigidities. However, if government debt has a longer maturity, $0 < \rho < 1$, adjustments in the ex post real return can be engineered via changes in the bond price P_t^M, which depends on inflation in future periods. This means that changes in the real debt return can be produced by a small, but sustained inflation, which is less costly than equivalent large fluctuations in current inflation. As a result, long-term debt helps the policy maker achieve the desired adjustment in the ex post real return at a smaller cost.

That completes the description of our model which consists of the usual resource constraint, consumption Euler equation and New Keynesian Phillips curve as well as the government's budget constraint and the bond pricing equation for longer-term bonds. These equations and the debt-dependent steady state are described in the Appendix C.1.

[8] We shall re-introduce short-term debt alongside longer-term debt in section 5 below.

3 First-Best Allocation

In some analyses of optimal fiscal policy (e.g., Aiyagari et al., 2002), it is desirable for the policy maker to accumulate a 'war chest' which pays for government consumption and/or fiscal subsidies to correct for other market imperfections. In order to assess to what extent our optimal, but time-consistent policy attempts to do so, it is helpful to define the level of government accumulated assets that would be necessary to mimic the social planner's allocation under the decentralized solution. The first step in doing so is defining the first-best allocation that would be implemented by the social planner. The social planner ignores the nominal inertia and all other inefficiencies, and chooses real allocations that maximize the representative consumer's utility, subject to the aggregate resource constraint and the aggregate production function. That is, the first-best allocation $\{C_t^*, N_t^*, G_t^*\}$ is the one that maximizes utility (36), subject to the technology constraint (35), and aggregate resource constraint $Y_t = C_t + G_t$.

The first order conditions imply that

$$(C_t^*)^{-\sigma} = \chi \left(G_t^*\right)^{-\sigma_g} = \left(N_t{}^*\right)^{\varphi} / A_t = \left(Y_t{}^*\right)^{\varphi} A_t^{-(1+\varphi)}$$

That is, given the resource constraints, it is optimal to equate the marginal utility of private and public consumption to the marginal disutility of labor effort and the optimal share of government consumption in output is

$$\frac{G_t^*}{Y_t^*} = \chi^{\frac{1}{\sigma_g}} \left(\frac{Y_t^*}{A_t}\right)^{-\frac{\varphi+\sigma_g}{\sigma_g}} A_t^{\frac{1-\sigma_g}{\sigma_g}}$$

In steady state (technology level A normalized to unity) and assuming $\sigma = \sigma_g$, this implies the optimal share of government consumption in output is

$$\frac{G^*}{Y^*} = \left(1 + \chi^{-\frac{1}{\sigma}}\right)^{-1}$$

and the first-best level of output is given by,

$$(Y^*)^{\varphi+\sigma} \left(1 - \frac{G^*}{Y^*}\right)^{\sigma} = 1 \tag{13}$$

It is illuminating to contrast the allocation achieved in the steady state of the decentralized equilibrium with this first best allocation. We do this by finding policies and prices that make the first-best allocation and the decentralized equilibrium coincide. Appendix C.1 shows that the steady-state level of output in the decentralized economy is given by,

$$Y^{\varphi+\sigma} \left(1 - \frac{G}{Y}\right)^{\sigma} = (1-\tau) \left(\frac{\epsilon - 1}{\epsilon}\right) \tag{14}$$

Comparing (14) and (13), and assuming the steady state share of government consumption is the same, then the two allocations will be identical when the labor income tax rate is set optimally to be,

$$\tau^* = 1 - \frac{\epsilon}{\epsilon - 1} = \frac{-1}{\epsilon - 1} \tag{15}$$

Notice that the optimal tax rate is negative, that is, it is effectively a subsidy which offsets the monopolistic competition distortion. This, in turn, requires that the government has accumulated a stock of assets defined as,

$$\frac{P^{M*}b^*}{4Y^*} = \frac{\beta}{4(1-\beta)}\left[\frac{-1}{\epsilon} - \left(1+\chi^{-\frac{1}{\sigma}}\right)^{-1}\right]$$

Using our benchmark calibration below, this would imply that a stock of assets of 843.75% of GDP would be required to generate sufficient income to pay for government consumption and a labor income subsidy which completely offsets the effects of the monopolistic competition distortion. We shall see that the steady-state level of debt in our optimal policy problem while negative, falls far short of this 'war chest' value.

4 Optimal Policy Under Discretion

We assume that the policymaker cannot credibly commit to particular future policy actions. Instead, the policymaker reoptimizes his policy response each period, that is, this policy is time-consistent. In our model, the presence of government debt makes the optimal time-consistent policy history dependent, in that the future path of the policy instruments depends on today's level of government debt.

The policy under discretion can be described as a set of decision rules for $\{C_t, Y_t, \Pi_t, b_t, \tau_t, G_t\}$ which maximize,

$$V(b_{t-1}, A_t) = \max\left\{\frac{C_t^{1-\sigma}}{1-\sigma} + \chi\frac{G_t^{1-\sigma_g}}{1-\sigma_g} - \frac{(Y_t/A_t)^{1+\varphi}}{1+\varphi} + \beta E_t\left[V(b_t, A_{t+1})\right]\right\}$$

subject to the resource constraint (32), the New Keynesian Phillips curve (33), and the government's budget constraint,

$$\begin{aligned}
&\beta E_t\left\{\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\left(\frac{P_t}{P_{t+1}}\right)\left(1+\rho P_{t+1}^M\right)\right\}b_t \qquad (16)\\
&= \left(1+\rho\beta E_t\left\{\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\left(\frac{P_t}{P_{t+1}}\right)\left(1+\rho P_{t+1}^M\right)\right\}\right)\frac{b_{t-1}}{\Pi_t}\\
&- \left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{Y_t}{A_t}\right)^{1+\varphi}C_t^{\sigma} + G_t
\end{aligned}$$

where we have used the bond pricing equation (31) to eliminate the current value of the bond in (34).

Defining auxiliary functions,

$$M(b_t, A_{t+1}) = (C_{t+1})^{-\sigma}Y_{t+1}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*} - 1\right)$$

$$L(b_t, A_{t+1}) = (C_{t+1})^{-\sigma}(\Pi_{t+1})^{-1}(1+\rho P_{t+1}^M)$$

we can write the constraints (33) and (16) facing the policy maker as,

$$(1-\epsilon) + \epsilon(1-\tau_t)^{-1}Y_t^{\varphi}C_t^{\sigma}A_t^{-1-\varphi} - \phi\frac{\Pi_t}{\Pi^*}\left(\frac{\Pi_t}{\Pi^*} - 1\right) + \phi\beta C_t^{\sigma}Y_t^{-1}E_t\left[M(b_t, A_{t+1})\right] = 0 \quad (17)$$

$$0 = \beta b_t C_t^{\sigma} E_t\left[L(b_t, A_{t+1})\right] - \frac{b_{t-1}}{\Pi_t}\left(1 + \rho\beta C_t^{\sigma} E_t\left[L(b_t, A_{t+1})\right]\right) + \left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{Y_t}{A_t}\right)^{1+\varphi} C_t^{\sigma} - G_t \tag{18}$$

By using the auxiliary functions in this way, we take account of the fact that the policy maker recognizes the impact their actions on the endogenous state, but that they cannot commit to future policy actions beyond that. In other words, we have a time-consistent policy. Therefore, the Lagrangian for the policy problem can be written as,

$$\begin{aligned}
\mathcal{L} = & \left\{\frac{C_t^{1-\sigma}}{1-\sigma} + \chi\frac{G_t^{1-\sigma_g}}{1-\sigma_g} - \frac{(Y_t/A_t)^{1+\varphi}}{1+\varphi} + \beta E_t[V(b_t, A_{t+1})]\right\} \\
& + \lambda_{1t}\left[Y_t\left(1 - \frac{\phi}{2}\left(\frac{\Pi_t}{\Pi^*} - 1\right)^2\right) - C_t - G_t\right] \\
& + \lambda_{2t}\left[\begin{array}{c} (1-\epsilon) + \epsilon(1-\tau_t)^{-1}Y_t^{\varphi}C_t^{\sigma}A_t^{-1-\varphi} - \phi\frac{\Pi_t}{\Pi^*}\left(\frac{\Pi_t}{\Pi^*} - 1\right) \\ +\phi\beta C_t^{\sigma}Y_t^{-1}E_t\left[M(b_t, A_{t+1})\right] \end{array}\right] \\
& + \lambda_{3t}\left[\begin{array}{c} \beta b_t C_t^{\sigma}E_t\left[L(b_t, A_{t+1})\right] - \frac{b_{t-1}}{\Pi_t}\left(1 + \rho\beta C_t^{\sigma}E_t\left[L(b_t, A_{t+1})\right]\right) \\ +\left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{Y_t}{A_t}\right)^{1+\varphi} C_t^{\sigma} - G_t \end{array}\right]
\end{aligned}$$

We can write the first order conditions (FOCs) for the policy problem as follows:

Consumption,

$$\begin{aligned}
& C_t^{-\sigma} - \lambda_{1t} + \lambda_{2t}\left[\sigma\epsilon(1-\tau_t)^{-1}Y_t^{\varphi}C_t^{\sigma-1}A_t^{-1-\varphi} + \sigma\phi\beta C_t^{\sigma-1}Y_t^{-1}E_t\left[M(b_t, A_{t+1})\right]\right] \\
& + \lambda_{3t}\left[\begin{array}{c} \sigma\beta b_t C_t^{\sigma-1}E_t\left[L(b_t, A_{t+1})\right] - \rho\sigma\beta\frac{b_{t-1}}{\Pi_t}C_t^{\sigma-1}E_t\left[L(b_t, A_{t+1})\right] \\ +\sigma\left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{Y_t}{A_t}\right)^{1+\varphi} C_t^{\sigma-1} \end{array}\right] = 0
\end{aligned} \tag{19}$$

which says that higher consumption increases utility, tightens the resource constraint ($\lambda_{1t} \geq 0$), has adverse effects on the inflation-output trade-offs at time t ($\lambda_{2t} \leq 0$), and relaxes the government budget constraint ($\lambda_{3t} \geq 0$);

Government spending,

$$\chi G_t^{-\sigma_g} - \lambda_{1t} - \lambda_{3t} = 0 \tag{20}$$

which says that higher government spending increases utility, tightens the resource constraint ($\lambda_{1t} \geq 0$), and tightens the government budget constraint ($\lambda_{3t} \geq 0$);

Output,

$$\begin{aligned}
& -Y_t^{\varphi}A_t^{-1-\varphi} + \lambda_{1t}\left[1 - \frac{\phi}{2}\left(\frac{\Pi_t}{\Pi^*} - 1\right)^2\right] \\
& +\lambda_{2t}\left[\epsilon\varphi(1-\tau_t)^{-1}Y_t^{\varphi-1}C_t^{\sigma}A_t^{-1-\varphi} - \phi\beta C_t^{\sigma}Y_t^{-2}E_t\left[M(b_t, A_{t+1})\right]\right] \\
& + \lambda_{3t}\left[(1+\varphi)Y_t^{\varphi}C_t^{\sigma}\left(\frac{\tau_t}{1-\tau_t}\right)A_t^{-1-\varphi}\right] = 0
\end{aligned} \tag{21}$$

which says that higher output (requiring higher labor) decreases utility, relaxes the resource constraint ($\lambda_{1t} \geq 0$), has adverse effects on the inflation-output trade-offs at time t ($\lambda_{2t} \leq 0$), and relaxes the government budget constraint ($\lambda_{3t} \geq 0$);

Taxation,

$$\lambda_{2t}\left[\epsilon(1-\tau_t)^{-2}Y_t^{\varphi}C_t^{\sigma}A_t^{-1-\varphi}\right]+\lambda_{3t}\left[Y_t^{1+\varphi}C_t^{\sigma}(1-\tau_t)^{-2}A_t^{-1-\varphi}\right]=0$$

simplifying,

$$\epsilon\lambda_{2t}+\lambda_{3t}Y_t=0 \tag{22}$$

which says that higher tax rate has adverse effects on the inflation-output trade-off at time t ($\lambda_{2t}\leq 0$), but relaxes the government budget constraint ($\lambda_{3t}\geq 0$);

Inflation,

$$\begin{aligned}-\lambda_{1t}\left[Y_t\frac{\phi}{\Pi^*}\left(\frac{\Pi_t}{\Pi^*}-1\right)\right]-\lambda_{2t}\left[\frac{\phi}{\Pi^*}\left(\frac{2\Pi_t}{\Pi^*}-1\right)\right]\\+\lambda_{3t}\left[\frac{b_{t-1}}{\Pi_t^2}\left(1+\rho\beta C_t^{\sigma}E_t\left[L(b_t,A_{t+1})\right]\right)\right]=0\end{aligned} \tag{23}$$

which says that higher inflation rate tightens the resource constraint ($\lambda_{1t}\geq 0$), has positive effects on the inflation-output trade-off at time t ($\lambda_{2t}\leq 0$), and relaxes the government budget constraint ($\lambda_{3t}\geq 0$);

Government debt,

$$\beta E_t[V_1(b_t,A_{t+1})]+\lambda_{2t}\left[\phi\beta C_t^{\sigma}Y_t^{-1}E_t\left[M_1(b_t,A_{t+1})\right]\right]$$

$$+\beta\lambda_{3t}\left[C_t^{\sigma}E_t\left[L(b_t,A_{t+1})\right]+b_tC_t^{\sigma}E_t\left[L_1(b_t,A_{t+1})\right]-\rho\frac{b_{t-1}}{\Pi_t}C_t^{\sigma}E_t\left[L_1(b_t,A_{t+1})\right]\right]=0$$

where $X_1(b_t,A_{t+1})\equiv\partial X(b_t,A_{t+1})/\partial b_t$ for the functions, $X=\{V,L,M\}$. Note that by the envelope theorem,

$$V_1(b_{t-1},A_t)=-\lambda_{3t}\frac{1}{\Pi_t}\left(1+\rho\beta C_t^{\sigma}E_t\left[L(b_t,A_{t+1})\right]\right)$$

we can write the FOC for government debt as,

$$\begin{aligned}0=-\beta E_t\left[\lambda_{3t+1}\frac{1}{\Pi_{t+1}}(1+\rho\beta C_{t+1}^{\sigma}E_{t+1}\left[L(b_{t+1},A_{t+2})\right])\right]+\lambda_{2t}\left[\phi\beta C_t^{\sigma}Y_t^{-1}E_t\left[M_1(b_t,A_{t+1})\right]\right]\\+\beta\lambda_{3t}\left[C_t^{\sigma}E_t\left[L(b_t,A_{t+1})\right]+b_tC_t^{\sigma}E_t\left[L_1(b_t,A_{t+1})\right]-\rho\frac{b_{t-1}}{\Pi_t}C_t^{\sigma}E_t\left[L_1(b_t,A_{t+1})\right]\right]\end{aligned} \tag{24}$$

The discretionary equilibrium is determined by the system given by the FOCs, (19), (20), (21), (22), (23), (24), and the constraints, (32), (17) and (18), and finally the exogenous process for the technology shock,

$$a_t=\rho_a a_{t-1}+e_{at}$$

where $a_t=\ln A_t$, and $e_{at}\overset{i.i.d}{\sim}N(0,\sigma_a^2)$.

Note there is a two period ahead expectation implicit in (24), related to the forward pricing of future longer term bonds. Using (22) and the definition of bond prices this can be simplified as,

$$\underbrace{\beta\lambda_{3t}C_t^{\sigma}E_t\left[L(b_t,A_{t+1})\right]-\beta E_t\left[\frac{\lambda_{3t+1}}{\Pi_{t+1}}(1+\rho P_{t+1}^{M})\right]}_{\text{trade-off between current and future distortions}}$$

$$\underbrace{-\lambda_{3t}\phi\epsilon^{-1}\beta C_t^{\sigma}E_t\left[M_1(b_t,A_{t+1})\right]+\beta\lambda_{3t}\left[b_tC_t^{\sigma}E_t\left[L_1(b_t,A_{t+1})\right]-\rho\frac{b_{t-1}}{\Pi_t}C_t^{\sigma}E_t\left[L_1(b_t,A_{t+1})\right]\right]}_{\text{additional terms due to lack of commitment}}=0 \tag{25}$$

since (7) implies that

$$P_t^M=\beta C_t^{\sigma}E_t\left[L(b_t,A_{t+1})\right] \tag{26}$$

Note that (25) is a generalized Euler equation, which involves the derivatives of the equilibrium policy rules with respect to the state variable, the stock of government debt. The standard trade-off between current and future distortions, reflected in the relationship between λ_{3t} and λ_{3t+1} in the first line of this expression, is actually a version of the tax-smoothing argument in Barro (1979), requiring that the marginal costs of taxation are equalized over time. This first line would drive the usual random walk in debt result, if the policy maker could commit. However, the presence of partial derivatives of debt in the second line is due to the time-consistency requirement, which reflects the fact that the future government cannot commit to future policy actions, but can affect the future through the level of debt it bequeaths to tomorrow. The first term on the second line reflects the fact that inflation expectations rise with debt levels (through the inflation and debt stabilization biases discussed elsewhere in the paper), $M_1(b_t, A_{t+1}) > 0$, and since this is costly in the presence of nominal inertia, there is a desire to deviate from tax smoothing, in order to reduce debt and the associated increase in inflation.

The second term in square brackets in the second line captures the impact of higher debt on bond prices. Since higher debt raises inflation, which in turn reduces bond prices, $L_1(b_t, A_{t+1}) < 0$, this term also serves to encourage a reduction in debt levels, when debt is relatively short-term. The magnitude of this effect is reduced as we increase the average maturity of government debt, ρ, and may even be reversed if term in square brackets turns positive as ρ is increased. Effectively, the lower bond prices mean we need to issue more bonds to finance a given deficit, but pay less to buy-back the existing debt stock. As debt maturity is increased, the latter effect rises relative to the former, and hence the desire to reduce debt levels is reduced, ceteris paribus. This trade-off between tax-smoothing and the time-consistency problems determines the equilibrium level of debt and inflation, where we expect inflation to be lower as debt maturity rises, for a given level of debt.

We can solve the nonlinear system consisting of these six first order conditions, the three constraints and (26) to yield the time-consistent optimal policy. Specifically, we need to find these ten time-invariant Markov-perfect equilibrium policy rules which are functions of the two state variables $\{b_{t-1}, A_t\}$.

5 Numerical Analysis

5.1 Solution Method and Calibration

For the model described in the previous section, the equilibrium policy functions cannot be computed in closed form. We thus resort to computational methods and derive numerical approximations to the policy rules. Local approximation methods are not applicable for this purpose, because the model's steady state around which local dynamics should be approximated is endogenously determined as part of the model solution and thus a priori unknown. In light of this difficulty, we resort to a global solution method. Specifically, we use Chebyshev collocation with time iteration to solve the model.[9] The detailed algorithm is presented in section C.2 in the appendix. In general, optimal discretionary policy problems can be characterized as a dynamic game between the private sector and successive governments. Multiplicity of equilibria is a common problem in dynamic games. One strategy has been to focus on equilibria with continuous strategies, see Judd (2004) for a discussion. Since we use polynomial approximations, we were searching only for continuous Markov-perfect equilibria where agents condition their strategies on payoff-relevant state variables.

Before solving the model numerically, the benchmark values of structural parameters must be specified. The calibration of parameters is summarized in Table 1. We set $\beta = (1/1.02)^{1/4} = 0.995$, which is a standard value for models with quarterly data and implies a 2% annual real interest rate. The intertemporal elasticity of substitution is set to one half ($\sigma = \sigma^g = 2$) which is in the middle of the parameter range typically considered in the literature. Labor supply elasticity is set to $\varphi^{-1} = 1/3$. The elasticity of substitution between intermediate goods is chosen as $\epsilon = 21$, which implies a monopolistic markup of approximately 5%, similar to Siu (2004). The coupon decay parameter, $\rho = 0.95$, corresponds to $4 \sim 5$ years of debt maturity, consistent with US data. The scaling parameter $\chi = 0.055$ ensures that the share of government spending in output is about 19%. The technology parameters are set to $\rho_a = 0.95$ and $\sigma_a = 0.01$. The price adjustment cost parameter $\phi = 32.5$ - implying, given the equivalence between the linearized NKPCs under Rotemberg and Calvo pricing (see Leith and Liu, 2014), that on average firms re-optimize prices every four to six months - is in line with empirical evidence. Finally, the annualized inflation target is chosen to be 2%, which is the current target adopted by most inflation targeting economies.

With this benchmark parameterization, we solve the fully nonlinear models via the Chebyshev collocation method. The maximum Euler equation error over the full range of the grid is of the order of 10^{-6}. As suggested by Judd (1998) , this order of accuracy is reasonable.

5.2 Numerical Results

In this section, we explore the properties of the equilibrium under the optimal time-consistent policy. We begin by considering the steady-state under our benchmark calibration, before turning to the transitional dynamics which highlights the state-dependent nature of the optimal policy mix. We then turn to consider the role of debt maturity in these results, highlighting the impact of debt maturity on the inflationary bias problem and the sensitivity of the policy to the level of government debt. We then allow the policy making to issue both short and longer-term bonds, and show that this enables the policy maker to import some of the policy mix associated with short-term debt, even though the bulk of its debt portfolio is longer-term debt. We conclude by exploring the implications of the policy maker suffering from a degree of myopia as a proxy for the costs of political frictions associated with operating fiscal policy.

[9]See Judd (1998) for a textbook treatment of the involved numerical techniques.

5.2.1 Steady State

We begin by plotting the policy functions for our benchmark calibration, in order to assess the steady-state of our optimal policy problem. Figure 3 plots policy rules against lagged debt, where the grid for A_t is fixed at 1. The first subplot illustrates how to find the steady state debt associated with the time consistent equilibrium. We should note that long-run debt under the benchmark parameterization is negative (-0.15), but smaller in absolute value than the first best level (-2.47), implying a stock of assets of 63.34% of GDP rather than the 'war chest' value of 843.75% of GDP. Intuitively, the interaction of inflation and debt stabilization biases generate a small negative long-run optimal value for debt, which falls far short of the accumulated level of assets that would be needed to finance government consumption and eliminate tax and other distortions. In addition, the annualized inflation rate in the steady state is 1.25%, which is less than the target of 2%. That is, there is an undershooting of the inflation target in steady state. Table **??** summarizes the steady state values.

In standard analyses of the inflationary bias problem, the magnitude of the bias is determined by the exogenously given degree of monopolistic competition which implies that the equilibrium level of output is inefficiently low. In the presence of debt and distortionary taxation, at higher debt or tax levels, the inefficiency is more pronounced, and hence the desire to generate a surprise inflation is greater, ceteris paribus. In other words, the inflation bias problem is endogenous, since the inefficiencies associated with distortionary taxation are likely to vary with the level of debt. At the same time, any surprise inflation reduces the real value of government debt and mitigates the costs of distortionary taxation, and ultimately the associated inflationary bias in the future - we follow Leith and Wren-Lewis (2013) in labelling this the 'debt stabilization bias'. As a result, the policy maker will seek to reduce debt levels to mitigate the costs of distortionary taxation and the endogenous inflationary bias problem. However, once debt turns negative, the policy maker faces a trade-off. Any surprise inflation will boost output, moving it closer to the efficient level. However, when the government is holding a net stock of nominal assets rather than liabilities, any surprise inflation will reduce the real value of those assets, and thereby worsen the future inefficiencies in the economy. The steady-state then balances these opposing forces, such that there is a small stock of positive government assets and a mild deflationary bias beyond which the government is not tempted to induce further deflationary surprises. The reason is that this would worsen output levels in the short-run, even though they would lead to a greater stock of assets in the longer run. Alternatively, this can be seen by considering the steady-state solution to the first order condition for debt, (25), temporarily removing the technology shock to render the model deterministic,

$$\phi\epsilon^{-1}M_1(b,A) = b(1-\rho\frac{1}{\Pi})L_1(b,A) \tag{27}$$

Since higher debt levels raise inflation, $M_1(b,A) > 0$, and that inflation reduces bond prices, $L_1(b,A) < 0$, this equation can only hold with a negative stock of debt in steady-state. Moreover, the steady-state debt and inflation level it implies will be a non-linear function of debt maturity and the magnitude of the inflation and debt stabilization biases.

We can see this by considering Table 2, which confirms that debt maturity has a nonlinear effect on the steady state debt to GDP ratio and inflation rate. The steady-state time-consistent level of accumulated assets held by the government first increases, and then decreases, as the average maturity of debt lengthens. Correspondingly, there is an an overshooting of the inflation target for debt maturity undershooting of the inflation target becomes less severe initially, but deteriorates afterwards. The intuition can be understood as follows. As noted above the policy maker essentially faces two biases - the conventional inflation bias where the policy maker wishes to induce a surprise inflation to boost activity in a sub-optimally small economy, and a debt stabilization bias where the policy maker wishes to use surprise inflation to reduce the value of debt or increase the real value of its nominal assets. At low maturity levels, in steady state the inflationary bias dominates, such that inflation lies above its target value. As maturity levels rise slightly, the inflationary bias falls, since the government accumulates a larger stock of assets which support lower tax rates, even though government consumption as a share of GDP rises slightly. As maturity rises more, the debt stabilization bias starts to outweigh the inflation bias. As a result, steady state inflation lies below target, due to the stock of nominal assets the government has accumulated. These nominal asset stocks, along with falls in government consumption relative to GDP, help support the reduced tax rates. It should be noted that the movements in debt to GDP ratios, tax rates and the share of public consumption in output are not entirely monotonic as maturity levels change, reflecting the balancing of the two forms of bias and their associated impact on the policy mix differences emphasized in the following subsection. However, the overwhelming tendency is for the debt stabilization bias to prevent the policy maker from accumulating a war chest of nominal assets sufficient to finance all government activities. Especially, this is the case when debt is of a shorter maturity.

5.2.2 Transition Dynamics and the Policy Mix

Before plotting the transition dynamics, it is helpful to consider the non-linearities implied by the policy functions, as plotted in Figure 3. Here we can see that inflation is rising steeply with the level of debt, as the endogenous inflationary and debt stabilization biases worsen with rising debt levels. We can also see how the policy response varies with debt levels - as debt levels rise, we see reduced government consumption, higher tax rates, and since higher debt levels raise inflation, a rise in real interest rates as well. However, once debt levels rise sufficiently, we can see that the rise in labor income tax rates slows, and real interest rates start to fall. This suggests that we may start to see a change in the policy mix, as we transition from high levels of debt towards the steady state.

Figure 4 plots the transition dynamics starting from a high level of debt, given the benchmark calibration. Here we can see the non-linearities implicit in the policy functions plotted in Figure 3. At very high initial levels of debt, we have a massive inflationary/debt stabilization bias problem (with annualized inflation in excess of 40%), and as a result, the policy maker is acting to reduce the level of debt fairly rapidly. To do so, they cut government spending and raise labor income tax rates. As a result of the high inflation, they also raise real interest rates. This is in line with the conventional monetary and fiscal policy assignment - fiscal policy is stabilizing debt and monetary policy is raising real interest rates to reduce aggregate demand and, thereby, inflation. However, looking closely at the start of the transition when debt levels are particularly high, we see a different policy mix - real interest rates are rising in the first few periods, as inflation and debt fall. Essentially in the first few periods, debt levels are so high that monetary policy is moderated to mitigate the effects of raised debt service costs. We shall now show that these changes in the policy mix are highly dependent on the maturity structure.

5.2.3 The Role of Debt Maturity

To illustrate the importance of the maturity structure on the optimal policy mix, we plot the policy functions for inflation, real interest rates, and the labor income tax, as a function of debt levels for the conventional single period debt ($\rho = 0$) and longer maturity debt ($\rho = 0.7588$, 0.9598, 0.9786, or equivalently 1, 5 and 8 year debt, respectively), as shown in Figure 5. In the case of single period debt, we obtain a large endogenous inflationary/debt stabilization bias problem, but find that even when inflation is high as a result, real interest rates fall at higher debt levels. Moreover, although tax rates initially rise with the level of debt, they start to fall once the debt to GDP ratio passes 30%. Therefore, we find that real interest rates are lower as debt levels rise despite the associated rsie in inflation, since monetary policy seeks to reduce debt service costs and expand the tax base. This would look like a passive monetary policy, if one was to estimate a standard policy rule. At the same time, tax policy looks conventional at low to moderate debt levels, but once debt levels rise above 30% of GDP, higher debt is associated with lower tax rates in an attempt to moderate inflation - an apparently active fiscal policy.

When we turn to the longer maturity debt ($\rho = 0.9598$ for example), we have conventional policies in place for a wider range of debt to GDP ratios. As debt levels rise, we have a worsening of the inflationary/debt stabilization bias problems, although not as pronounced as in the case of shorter maturity debt, since the desire to reduce debt at any given positive debt level is lower. Note that suppressed bond prices reduce the costs of debt buy-back at longer maturities. However, unlike the case of single period debt, monetary policy raises real interest rates, in response to this rise in inflation until debt to GDP ratios exceed 175% of GDP at which point they start falling sharply, as debt levels rise further. At the same point, labor income tax rates start falling with rising debt levels, too. Therefore, we have a policy mix which looks like the conventional policy assignment at lower debt levels, that is, real interest rates rise to fight inflation, while tax rates increase and government consumption falls to stabilize debt. However, at higher debt levels, we observe a reversal in the policy mix, that is, monetary policy reduces real interest rates to stabilize debt, while fiscal policy moderates the increases in tax rates to mitigate the rise in inflation.

We can then see the role of debt maturity on the transition dynamics, by plotting the transition paths for four cases of ρ: $\rho = 0$ (single period debt), $\rho = 0.7588$ (1 year debt maturity), $\rho = 0.9598$ (5 year debt maturity), and $\rho = 0.9786$ (8 year debt maturity). We begin from the same debt to GDP ratio, as depicted in Figure 6. Here we can see the radically higher inflationary/debt stabilization bias problems when debt maturity is low, and the unconventional policy mix this engenders - real interest rates are cut to help reduce debt when debt is single period, tax increases are moderated and government consumption is markedly reduced. As debt maturity is increased, we both reduce the debt stabilization bias problem and the conventional policy mix is applied at lower debt to GDP ratios.

5.2.4 Endogenizing Debt Maturity

Up until this point, we have held the level of debt maturity fixed by controlling ρ. We now allow the policy maker to have some control over the maturity structure, by allowing them to issue a mixture of single period and longer-maturity debt of a given ρ. By varying the relative proportions of these two types of bonds, the policy maker can influence the average maturity of the outstanding stock of debt. We plot the transition dynamics for the benchmark calibration in Figure 7, where we start from the same initial overall debt to GDP ratio. Here it is important to stress that despite the high overall debt to GDP ratio, the quantity of short-term debt issued is very low. We do not observe the extreme portfolios made up of issuing long-term debt to purchase short-term assets. This portfolio has been used as a hedging device when policy makers can commit (see Debortoli et al. (2015) and Leeper and Leith (2017)). Instead, there is an extremely modest issuance of short-term debt, even when overall debt levels are very high. The short-term debt serves to support small changes in the time-consistent policy mix. Specifically, we do not observe significant changes in the paths for inflation or overall indebtedness, suggesting that the availability of short-term debt is not used to radically alter the speed of fiscal correction. Instead, the policy mix underpinning those dynamics does change - real interest rates, government consumption and tax rates are lower, when the policy maker can issue short-term debt and overall debt levels are high. In other words, the issuance of short-term debt tilts the policy mix towards the unconventional policies pursued at lower maturity levels, where more adjustment is borne by monetary policy and cuts in government spending, and less in tax increases. This tilting in the policy mix produces a very modest lifetime welfare gain (equivalent to 1.5% of one-period's steady-state consumption). If we turn to a lower maturity structure (an average medium-term debt maturity of two years), then the effects are qualitatively similar, but quantitatively much smaller - see Figure 8.

5.2.5 Fiscal Policy Myopia

One aspect of the equilibrium outcomes under time-consistent optimal policy is that the steady state level of debt is negative, capturing the balancing of the usual inflation bias with the debt stabilization bias. This is clearly at odds with the empirical fact that almost all advanced economy governments have issued net liabilities, rather than accumulated net assets.[10] One possible explanation for this is offered by the New Political Economy literature, which emphasizes that the political process may result in a deficit bias problem leading to an accumulation of public debt. While there are several explanations as to why such a bias may exist, these can loosely be introduced into our model by allowing the policy maker to be relatively myopic. To do so, we add a probability of survival, δ, to the policy maker's objective function,

$$E_0 \sum_{t=0}^{\infty} (\beta\delta)^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} + \chi \frac{G_t^{1-\sigma_g}}{1-\sigma_g} - \frac{{N_t}^{1+\varphi}}{1+\varphi} \right) \tag{28}$$

which implicitly assumes that when $\delta < 1$ they have a shorter time horizon than society as a whole. This implies that the policy maker does not fully appreciate the long-term benefits of reducing debt, but does care, relatively, about the short-run costs of doing so. The implications of this for the steady-state level of debt can be seen from considering the

[10] Some papers in the literature on time-consistent monetary and fiscal policies ensure a positive level of government debt in steady state via explicitly considering money. For example, Niemann et al. (2013) obtain this result through a carefully chosen money demand function, while Martin (2009) uses the cash-credit goods setup. Our paper is more in the spirit of conventional analyses of policy in New Keynesian models which typically assume a cashless economy.

deterministic steady-state of the first order condition for debt in the presence of myopia,

$$(1-\delta)\frac{1+\rho P^{m}}{\Pi}=\phi\epsilon^{-1}M_{1}(b,A)-b(1-\rho\frac{1}{\Pi})L_{1}(b,A) \tag{29}$$

Since the effects of debt, ceteris paribus, in raising inflation and lower bond prices imply $M_1(b, A) > 0$ and $L_1(b, A) < 0$, respectively, the steady state of this equation can be supported by a lower negative value of b, which can even turn positive as policy-maker myopia, δ, is reduced below 1.

Giving the policy maker a 25 year planning horizon implies $\delta = 0.99$ and results in the policy functions shown in Figure 9. Here we can see that the qualitative features of these policy functions are the same as previously - at high debt levels we get the switch in the policy mix from using fiscal policy to stabilize debt to relying on monetary policy. However, this switch in the policy mix occurs at an even higher level of debt than in the case with a non-myopic policy maker. Additionally, the steady state of the policy problem now involves both a positive debt to GDP ratio and an inflationary rather than deflationary bias. Essentially, the debt stabilization bias is reduced, as the policy maker is less inclined to incur the costs of debt reduction in order to achieve longer-term benefits. Policy maker myopia serves to render the equilibrium policy more plausible, in that inflation during the transition to the steady state is not as high as the case without myopia. In fact, the government stabilizes debt at a plausible level of 52.5% of GDP, with an associated inflation bias of 5.61%.[11]

6 Conclusions

In this paper we have considered the optimal monetary and fiscal policy mix in a New Keynesian economy with a plausible debt-maturity structure. The existence of nominal debt induces a substantial endogenous inflation and debt stabilization bias problem as the policy maker faces the temptation to both boost the economy and reduce the real value of debt through inflation surprises, respectively. In fact, under our benchmark calibration, this temptation results in a steady state where the government accumulates a small stock of assets (falling well short of the 'war chest' needed to finance all of the government's activities without recourse to distortionary taxation) and suffers a mild undershooting of the inflation target. Moreover, we find that the policy equilibrium is highly non-linear, depending crucially on both the level of debt and the maturity structure of that debt. Adopting single period debt implies a policy mix which can look quite unconventional, if debt levels rise above relatively modest levels. Specifically, monetary policy will seek to stabilize debt through lower debt interest payments, while tax policy attempts to stabilize inflation. With longer debt maturities, optimal policy looks more like the conventional policy assignment - monetary policy raises real interest rates to fight inflation, while taxes are raised to stabilize debt, unless debt level rise sufficiently high that we reverse the policy assignment as in the case of single period debt. This policy mix reversal occurs at far higher debt levels, as we move from single period debt to plausibly calibrated debt maturities.

We also consider the role of endogenous maturity by allowing the policy maker to issue

[11]It should be noted that we have assumed a inflation target of 2% where it is costless to adjust prices in line with that target, such that an inflation rate of 3.61% constitutes the costly deviation from that target.

both single-period and medium maturity debt. We find that this does little to affect the underlying inflation and debt stabilization bias problems and debt dynamics, but that a modest issuance of short-term debt allows the policy maker to shift the policy mix to be more like that of the single period debt case with lower real interest rates, government consumption and tax rates. This is mildly welfare improving. It is also interesting to note that the implicit government debt portfolio does not attempt to achieve any of the hedging effects associated with some optimal policy exercises when the policy maker can commit. Finally, we allow the policy maker to be relatively myopic in evaluating the future in a manner which mimics the various explanations of the deficit bias problem. We find that this does not qualitatively affect the debt-dependent bifurcation in the policy mix detailed in the paper, although it does for even a relatively modest degree of myopia turn the steady-state debt level positive and support an inflationary rather than deflationary bias, bringing us closer to understanding empirically observed debt dynamics.

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A Tables

Table 1: Parameterization

Parameter	Value	Definition
β	0.995	Quarterly discount factor
σ	2	Relative risk aversion coefficient
σ^g	2	Relative risk aversion coefficient for government spending
φ	3	Inverse Frish elasticity of labor supply
ϵ	21	Elasticity of substitution between varieties
ρ	0.95	Debt maturity structure
χ	0.055	Scaling parameter associated with government spending
ρ_a	0.95	AR-coefficient of technology shock
σ_a	0.01	Standard deviation of technology shock
ϕ	32.5	Rotemberg adjustment cost coefficient
Π^*	2%	Annual inflation rate target

Table 2: The steady states under alternative maturities

Parameter		Steady State Values							
maturity		real value of debt	debt-GDP ratio	government spending	consumption	annualized inflation rate	income tax rate	tax base	welfare
ρ	years	b	$\frac{100P^M b}{4Y}$	$\frac{G}{Y}$	$\frac{C}{Y}$	$100\,(\Pi^4 - 1)$	τ	wN	$\frac{V}{1-\beta}$
0	0.25	−0.74	−17.80%	19.94%	80.05%	2.53%	20.57%	0.7820	−352.1533
0.1	0.28	−0.79	−21.01%	19.96%	80.04%	2.44%	20.53%	0.7826	−352.1438
0.2	0.31	−0.85	−25.29%	19.99%	80.01%	2.32%	20.46%	0.7834	−352.1314
0.3	0.35	−0.92	−31.17%	20.01%	79.99%	2.17%	20.37%	0.7846	−352.1146
0.4	0.41	−1.00	−39.45%	20.04%	79.96%	1.98%	20.23%	0.7864	−352.0913
0.5	0.49	−1.08	−51.22%	20.06%	79.94%	1.72%	20.00%	0.7891	−352.0586
0.6	0.61	−1.13	−66.61%	20.06%	79.94%	1.40%	19.68%	0.7929	−352.0151
0.7	0.81	−1.01	−79.26%	19.99%	80.00%	1.11%	19.36%	0.7966	−351.9716
0.7588	1	−0.83	−80.87%	19.92%	80.07%	1.03%	19.24%	0.7977	−351.9551
0.8844	2	−0.36	−70.43%	19.67%	80.33%	1.12%	19.19%	0.7973	−351.9496
0.9263	3	−0.22	−65.10%	19.57%	80.42%	1.22%	19.20%	0.7968	−351.9550
0.9472	4	−0.16	−63.52%	19.53%	80.46%	1.24%	19.18%	0.7968	−351.9565
0.9598	5	−0.13	−63.59%	19.50%	80.49%	1.23%	19.15%	0.7971	−351.9555
0.9681	6	−0.11	−64.47%	19.49%	80.51%	1.20%	19.11%	0.7975	−351.9531
0.9741	7	−0.09	−66.24%	19.47%	80.52%	1.14%	19.05%	0.7981	−351.9493
0.9786	8	−0.08	−67.98%	19.46%	80.53%	1.08%	19.01%	0.7986	−351.9456
0.9821	9	−0.07	−70.27%	19.45%	80.54%	1.01%	18.95%	0.7992	−351.9412
0.9849	10	−0.07	−72.52%	19.45%	80.54%	0.94%	18.90%	0.7998	−351.9373

B Figures

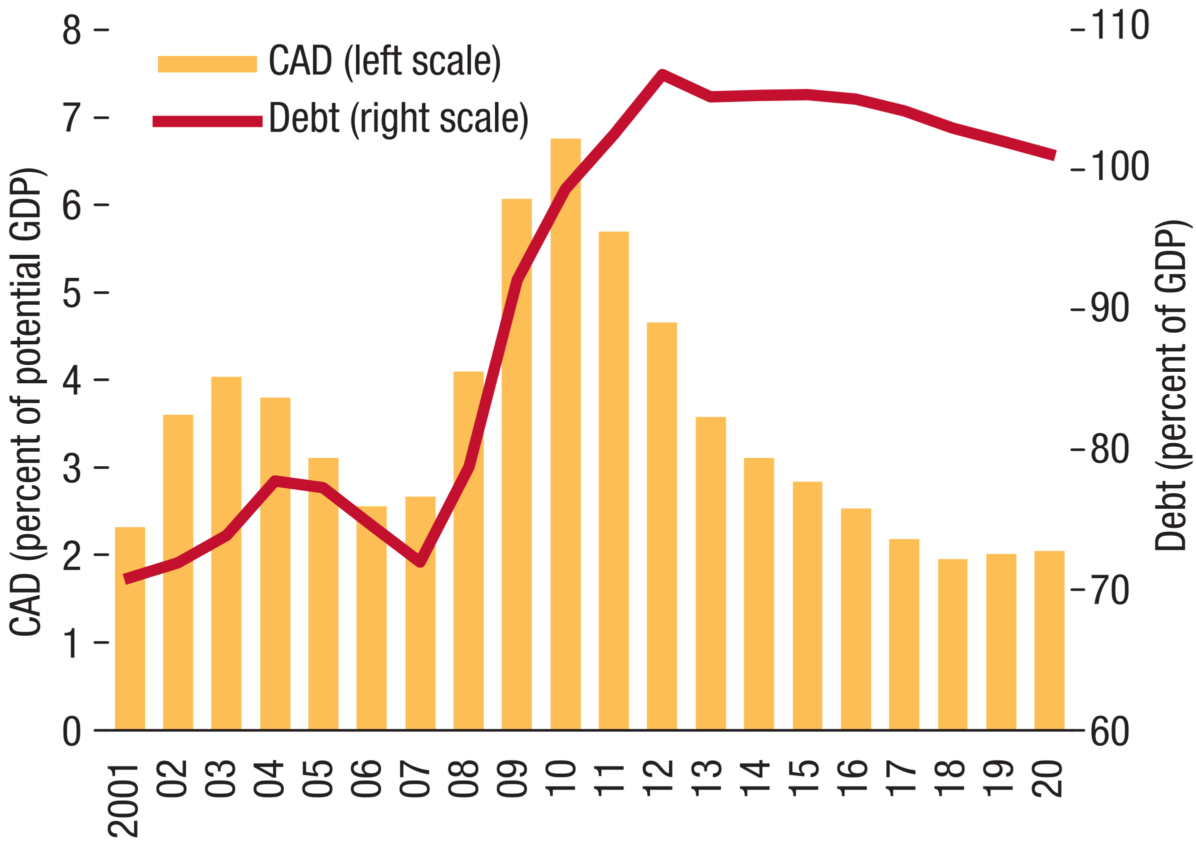

Figure 1: The average debt-GDP ratio and the cyclically adjusted deficit as percent of potential GDP in advanced economies. CAD = cyclically adjusted deficit. Source: IMF Fiscal Monitor, 2015.

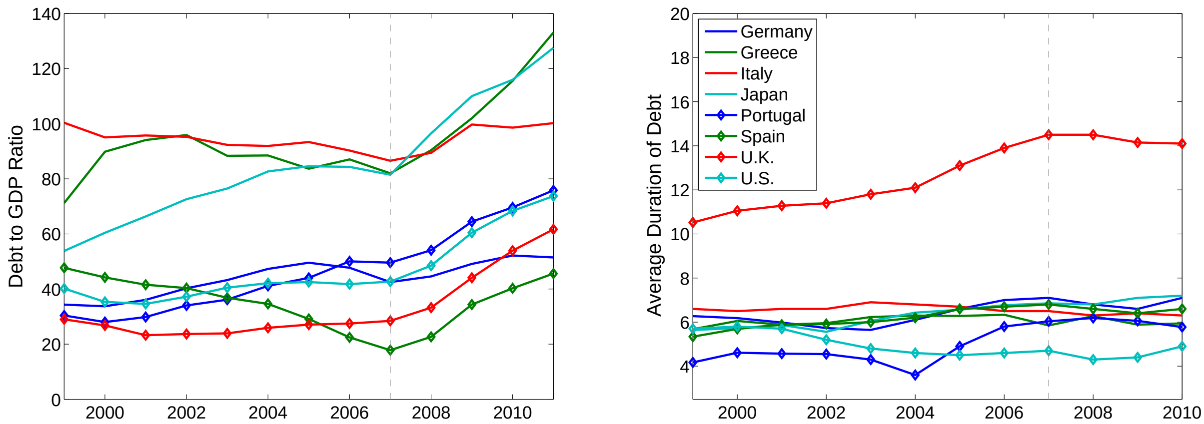

Figure 2: This figure shows the evolution of debt-to-GDP ratios and average maturity of debt for a selected group of countries. The debt-to-GDP time series is measured as net financial liabilities as a percentage of nominal GDP; the average maturity of debt is measured as the average term to maturity of total outstanding government debt. Taken from Eusepi and Preston (2013).

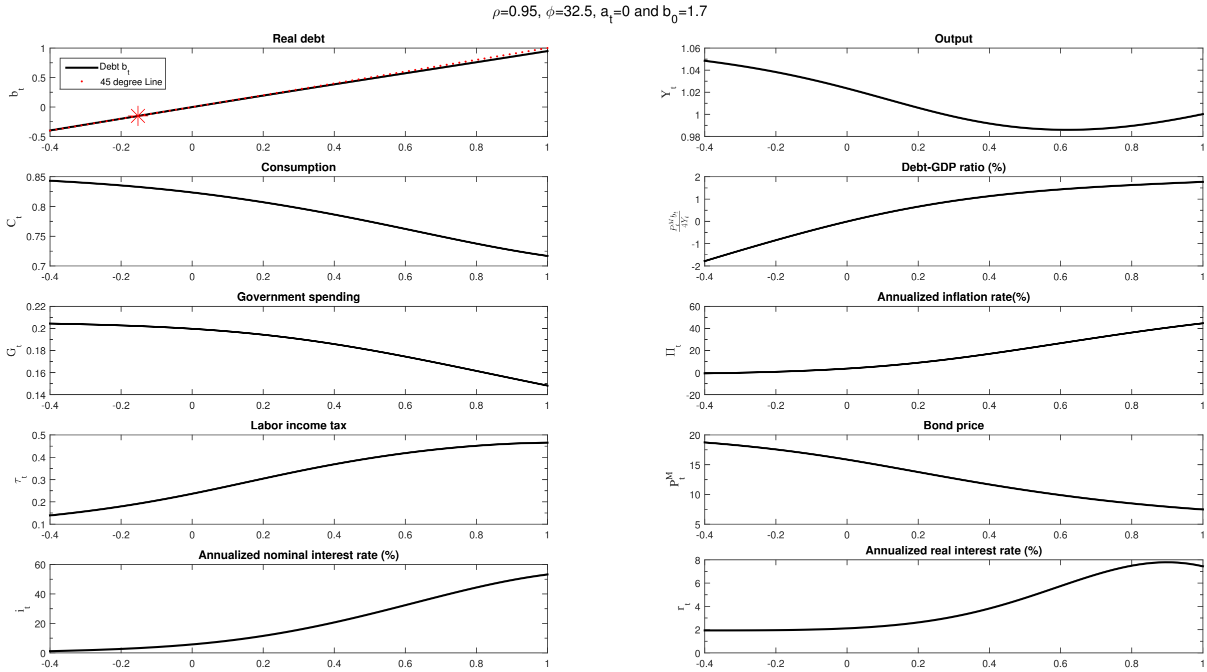

Figure 3: Under the benchmark parameters, the policy rules as functions of lagged debt, when the grid for technology is fixed at A=1. The cross sign indicates steady state.

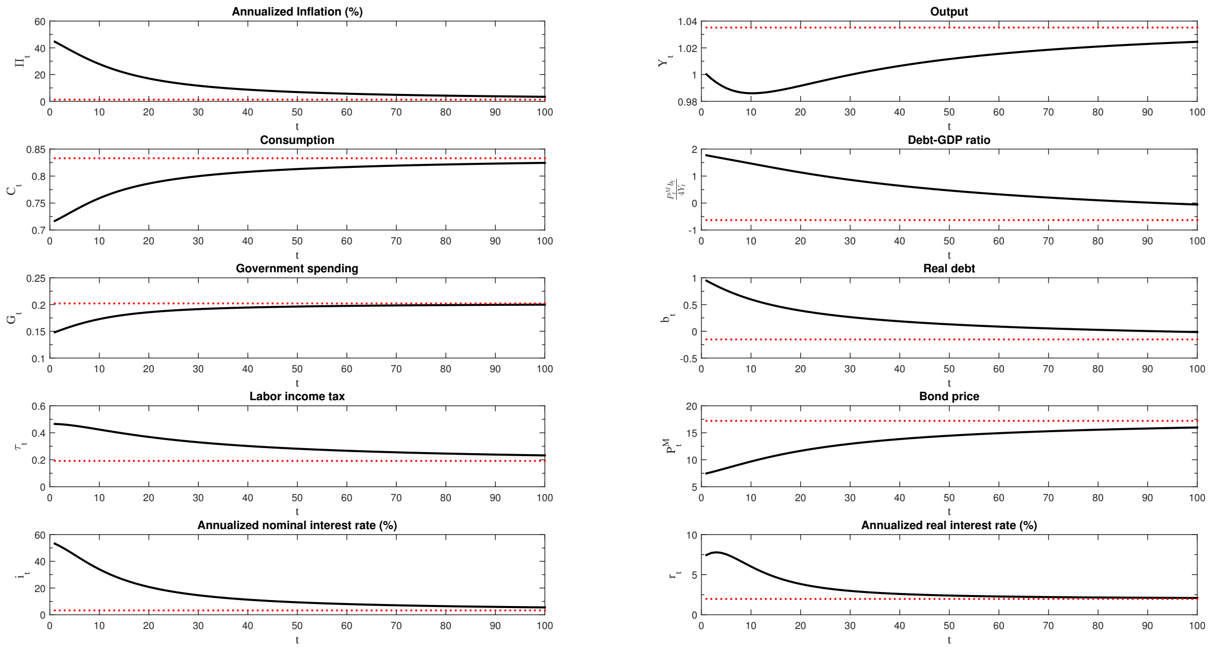

Figure 4: Under the benchmark parameters, this figure plots the transition paths of policy variables when debt starts from levels consistent with currently observed debt-GDP ratios, and technology is fixed at A=1. The red dotted lines indicate steady states.

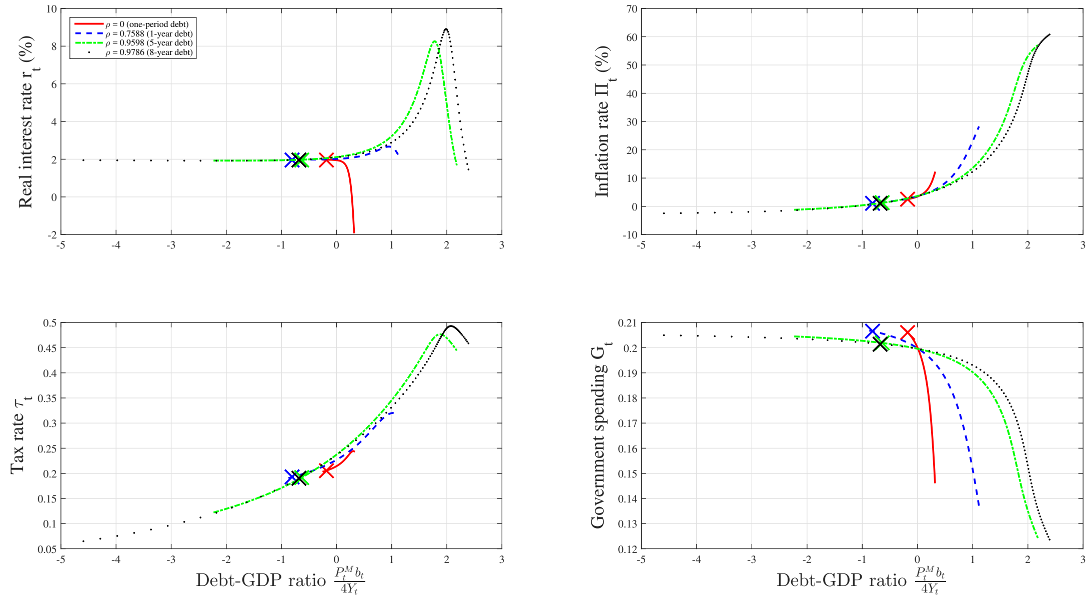

Figure 5: This figure illustrates the relationship between policy mix and the debt-GDP ratio under alternative maturities. The cross signs indicate steady states.

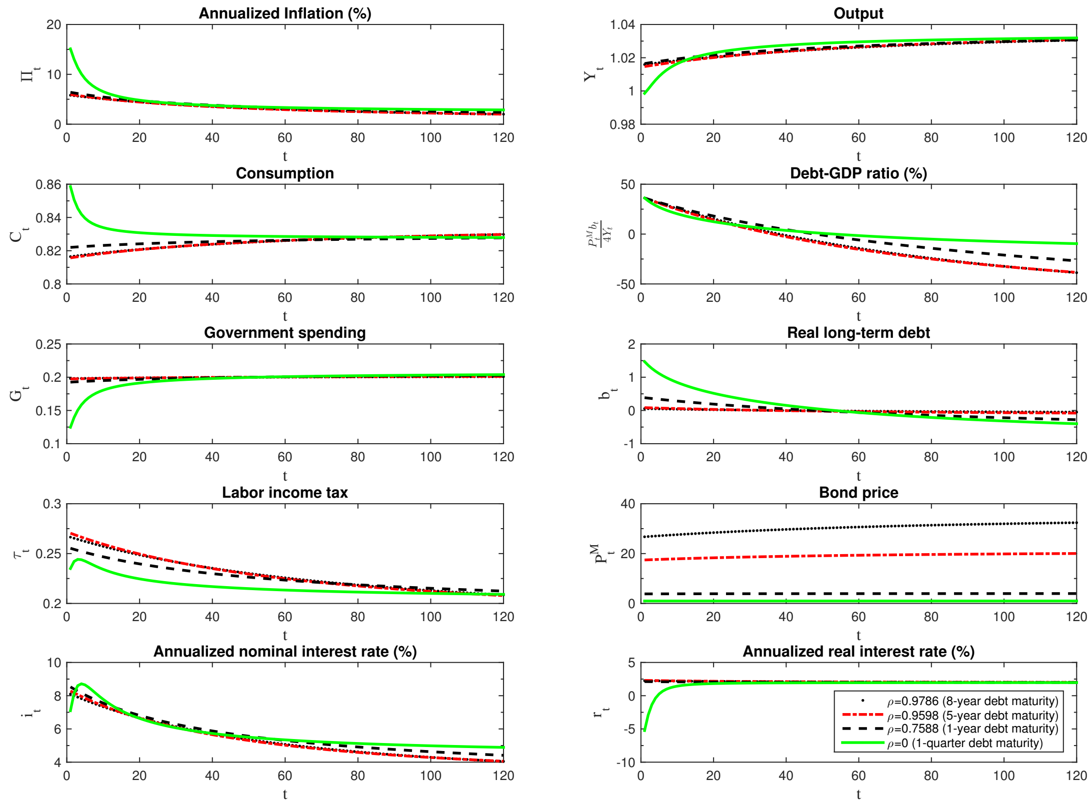

Figure 6: This figure plots the transition paths under different maturities, when the debt-GDP ratio starts from the same level.

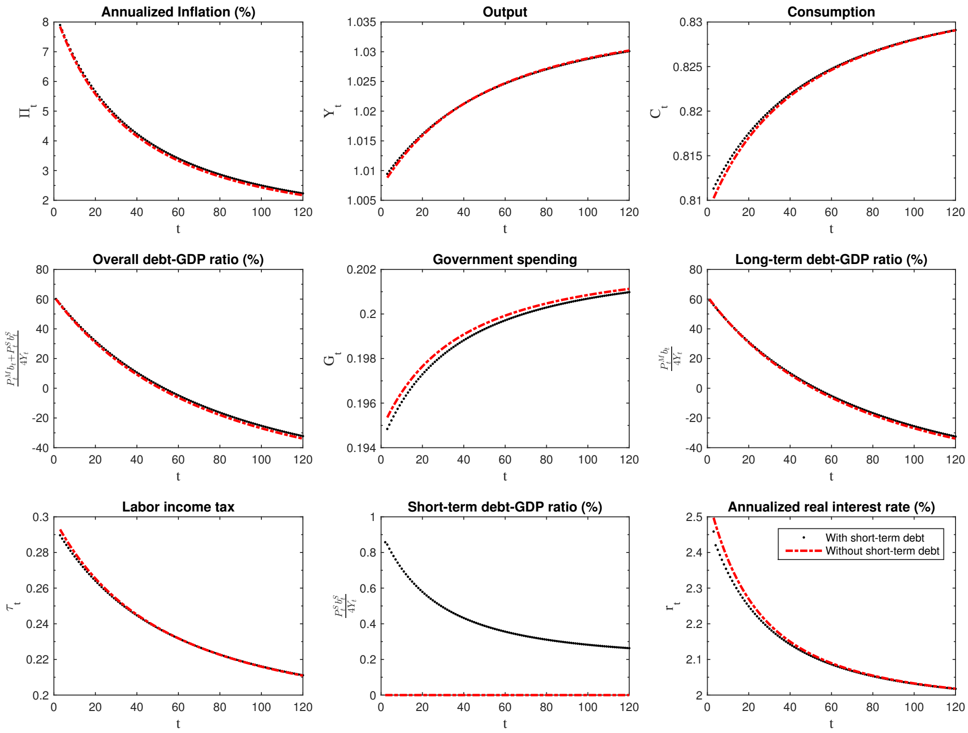

Figure 7: This figure compares the transition paths under the benchmark case with and without short-term debt, when the debt-GDP ratio starts from the same level.

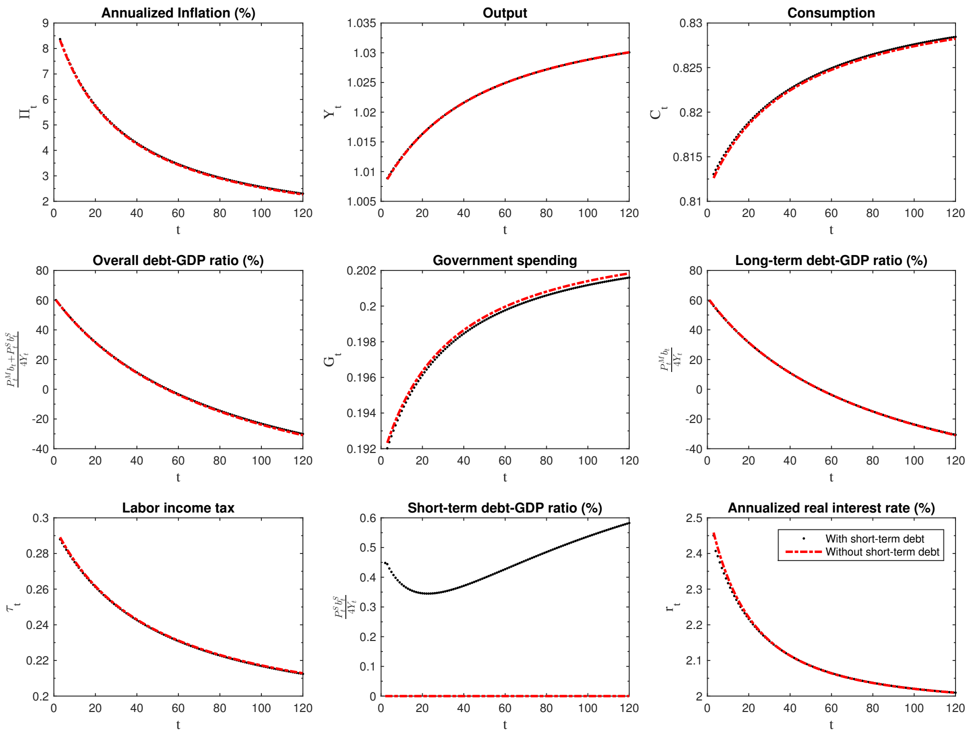

Figure 8: This figure compares the transition paths under the case with debt maturity of two years, with and without short-term debt, when the debt-GDP ratio starts from the same level.

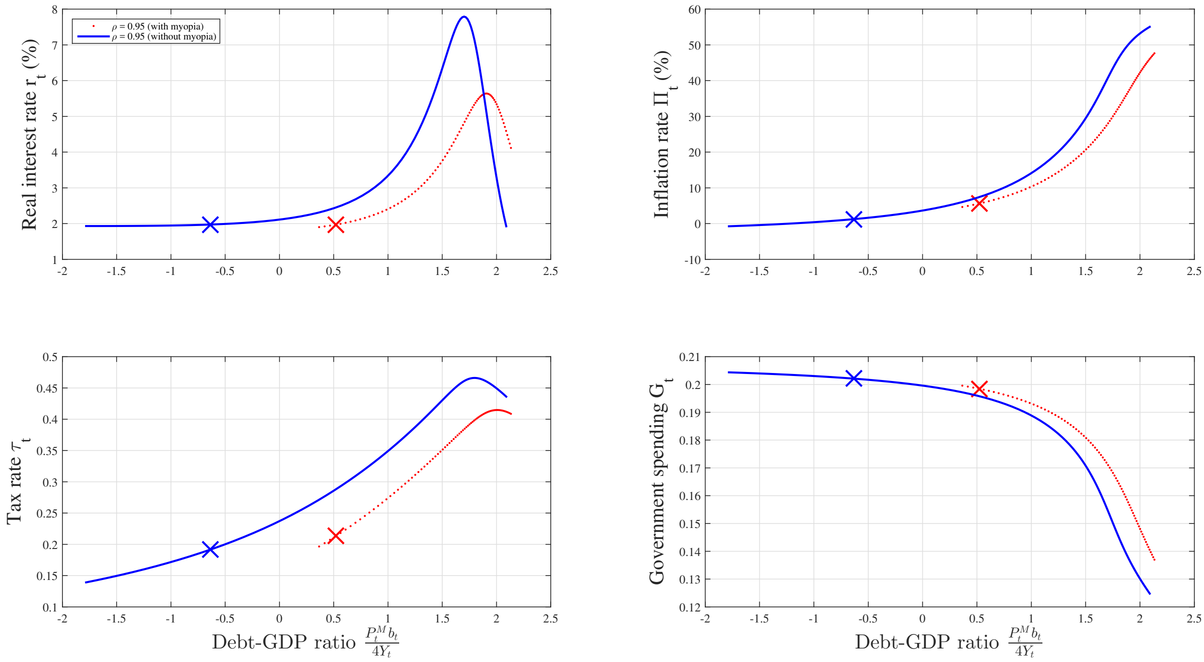

Figure 9: This figure illustrates the relationship between policy mix and the debt-GDP ratio under the benchmark case with and without myopia. The cross signs indicate steady states.

C Technical Appendix (Not for Publication)

C.1 Summary of Model

We now summerise the model and its steady state before turning to the time-consistent policy problem.

Consumption Euler equation,

$$\beta R_t E_t \left\{ \left(\frac{C_t}{C_{t+1}} \right)^{\sigma} \left(\frac{P_t}{P_{t+1}} \right) \right\} = 1 \tag{30}$$

Pricing of longer-term bonds,

$$\beta E_t \left\{ \left(\frac{C_t}{C_{t+1}} \right)^{\sigma} \left(\frac{P_t}{P_{t+1}} \right) \left(1 + \rho P_{t+1}^M\right) \right\} = P_t^M \tag{31}$$

Labour supply,

$$N_t^{\varphi} C_t^{\sigma} = (1 - \tau_t) \left(\frac{W_t}{P_t} \right) \equiv (1 - \tau_t) w_t$$

Resource constraint,

$$Y_t \left[1 - \frac{\phi}{2} \left(\frac{\Pi_t}{\Pi^*} - 1 \right)^2 \right] = C_t + G_t \tag{32}$$

Phillips curve,

$$\begin{aligned} 0 = (1 - \epsilon) + \epsilon mc_t - \phi \frac{\Pi_t}{\Pi^*} \left(\frac{\Pi_t}{\Pi^*} - 1 \right) \\ + \phi \beta E_t \left[\left(\frac{C_t}{C_{t+1}} \right)^{\sigma} \frac{Y_{t+1}}{Y_t} \frac{\Pi_{t+1}}{\Pi^*} \left(\frac{\Pi_{t+1}}{\Pi^*} - 1 \right) \right] \end{aligned} \tag{33}$$

Government budget constraint,

$$\begin{aligned} P_t^M b_t &= (1 + \rho P_t^M) \frac{b_{t-1}}{\Pi_t} - \frac{W_t}{P_t} N_t \tau_t + G_t \\ &= (1 + \rho P_t^M) \frac{b_{t-1}}{\Pi_t} - \left(\frac{\tau_t}{1 - \tau_t} \right) N_t^{1+\varphi} C_t^{\sigma} + G_t \\ &= (1 + \rho P_t^M) \frac{b_{t-1}}{\Pi_t} - \left(\frac{\tau_t}{1 - \tau_t} \right) \left(\frac{Y_t}{A_t} \right)^{1+\varphi} C_t^{\sigma} + G_t \end{aligned} \tag{34}$$

Technology,

$$Y_t = A_t N_t \tag{35}$$

Marginal costs,

$$mc_t = W_t / \left(P_t A_t\right) = (1 - \tau_t)^{-1} Y_t^{\varphi} C_t^{\sigma} A_t^{-1-\varphi}$$

The objective function for social welfare is given by,

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} + \chi \frac{G_t^{1-\sigma_g}}{1-\sigma_g} - \frac{(Y_t/A_t)^{1+\varphi}}{1+\varphi} \right) \tag{36}$$

There are two state variables, real debt b_t and productivity $a_t = \ln(A_t)$.

C.1.1 The Deterministic Steady State

Given the system of non-linear equations, the corresponding steady state system can be written as follows:

$$A = 1$$

$$\frac{\beta R}{\Pi} = 1$$

$$\frac{\beta}{\Pi}\left(1 + \rho P^M\right) = P^M$$

$$(1-\tau) w = N^{\varphi} C^{\sigma}$$

$$Y \left[1 - \frac{\phi}{2} \left(\frac{\Pi}{\Pi^*} - 1 \right)^2 \right] = C + G$$

$$(1-\epsilon) + \epsilon mc + \phi(\beta - 1) \left[\frac{\Pi}{\Pi^*} \left(\frac{\Pi}{\Pi^*} - 1 \right) \right] = 0$$

$$P^M b = (1 + \rho P^M) \frac{b}{\Pi} - \left(\frac{\tau}{1-\tau} \right) Y^{1+\varphi} C^{\sigma} + G$$

$$Y = N$$

$$mc = w = (1-\tau)^{-1} Y^{\varphi} C^{\sigma}$$

Hence, when $\Pi = \Pi^*$,

$$P^M = \frac{\beta}{\Pi^* - \beta\rho}$$

$$mc = w = \frac{\epsilon - 1}{\epsilon}$$

$$\frac{C}{Y} = \left[(1-\tau) \left(\frac{\epsilon - 1}{\epsilon} \right) \right]^{1/\sigma} Y^{-\frac{\varphi+\sigma}{\sigma}}$$

$$\frac{G}{Y} = 1 - \frac{C}{Y} = 1 - \left[(1-\tau) \left(\frac{\epsilon - 1}{\epsilon} \right) \right]^{1/\sigma} Y^{-\frac{\varphi+\sigma}{\sigma}}$$

$$P^M b = \frac{\beta}{1-\beta} \left[\tau \left(\frac{\epsilon - 1}{\epsilon} \right) - \frac{G}{Y} \right] Y$$

Note that,

$$Y^{\varphi+\sigma} \left(1 - \frac{G}{Y} \right)^{\sigma} = (1-\tau) \left(\frac{\epsilon - 1}{\epsilon} \right) \tag{37}$$

which will be used to contrast with the allocation that would be chosen by a social planner.

C.2 Numerical Algorithm

Let $s_t = (b_{t-1}, a_t)$ denote the state vector at time t, where real stock of debt b_{t-1} is endogenous and technology $A_t = \exp(a_t)$ is exogenous and respectively, with the following law of motion:

$$P_t^M b_t = (1+\rho P_t^M)\frac{b_{t-1}}{\Pi_t} - w_t N_t \tau_t + G_t$$

$$a_t = \rho_a a_{t-1} + e_{at}$$

where $0 \leq \rho_a < 1$ and technology innovation e_{at} is an *i.i.d.* normal random variable, which has a zero mean and a finite standard deviation σ_a.

There are 7 endogenous variables and 3 Lagrangian multipliers. Correspondingly, there are 10 functional equations associated with the 10 varaibles $\left\{C_t, Y_t, \Pi_t, b_t, \tau_t, P_t^M, G_t, \lambda_{1t}, \lambda_{2t}, \lambda_{3t}\right\}$. Let's define a new function $X : \mathbb{R}^2 \to \mathbb{R}^{10}$, in order to collect the policy functions of endogenous variables as follows:

$$X(s_t) = \left(C_t(s_t), Y_t(s_t), \Pi_t(s_t), b_t(s_t), \tau_t(s_t), P_t^M(s_t), G_t(s_t), \lambda_{1t}(s_t), \lambda_{2t}(s_t), \lambda_{3t}(s_t)\right)$$

Given the specification of the function X, the equilibrium conditions can be written more compactly as,

$$\Gamma\left(s_t, X(s_t), E_t\left[Z\left(X(s_{t+1})\right)\right], E_t\left[Z_b\left(X(s_{t+1})\right)\right]\right) = 0$$

where $\Gamma : \mathbb{R}^{2+10+3+3} \to \mathbb{R}^{10}$ summarizes the full set of dynamic equilibrium relationship, and

$$Z\left(X(s_{t+1})\right) = \begin{bmatrix} Z_1\left(X(s_{t+1})\right) \\ Z_2\left(X(s_{t+1})\right) \\ Z_3\left(X(s_{t+1})\right) \end{bmatrix} \equiv \begin{bmatrix} M(b_t, A_{t+1}) \\ L(b_t, A_{t+1}) \\ (\Pi_{t+1})^{-1}\left(1+\rho P_{t+1}^M\right)\lambda_{3t+1} \end{bmatrix}$$

with

$$M(b_t, A_{t+1}) = \left(C_{t+1}\right)^{-\sigma} Y_{t+1}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*} - 1\right)$$

$$L(b_t, A_{t+1}) = (C_{t+1})^{-\sigma}(\Pi_{t+1})^{-1}(1+\rho P_{t+1}^M)$$

and

$$Z_b\left(X(s_{t+1})\right) = \begin{bmatrix} \frac{\partial Z_1(X(s_{t+1}))}{\partial b_t} \\ \frac{\partial Z_2(X(s_{t+1}))}{\partial b_t} \\ \frac{\partial Z_3(X(s_{t+1}))}{\partial b_t} \end{bmatrix} \equiv \begin{bmatrix} \frac{\partial M(b_t, A_{t+1})}{\partial b_t} \\ \frac{\partial L(b_t, A_{t+1})}{\partial b_t} \\ \frac{\partial\left[(\Pi_{t+1})^{-1}\left(1+\rho P_{t+1}^M\right)\lambda_{3t+1}\right]}{\partial b_t} \end{bmatrix}$$

More specifically,

$$L_1(b_t, A_{t+1}) = \frac{\partial\left[(C_{t+1})^{-\sigma}(\Pi_{t+1})^{-1}(1+\rho P_{t+1}^M)\right]}{\partial b_t}$$

$$= -\sigma(C_{t+1})^{-\sigma-1}(\Pi_{t+1})^{-1}(1+\rho P_{t+1}^M)\frac{\partial C_{t+1}}{\partial b_t}$$
$$-(C_{t+1})^{-\sigma}(\Pi_{t+1})^{-2}(1+\rho P_{t+1}^M)\frac{\partial \Pi_{t+1}}{\partial b_t} + \rho(C_{t+1})^{-\sigma}(\Pi_{t+1})^{-1}\frac{\partial P_{t+1}^M}{\partial b_t}$$

and

$$M_1(b_t, A_{t+1}) = \frac{\partial\left[\left(C_{t+1}\right)^{-\sigma} Y_{t+1}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*} - 1\right)\right]}{\partial b_t}$$

$$= -\sigma(C_{t+1})^{-\sigma-1}Y_{t+1}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*}-1\right)\frac{\partial C_{t+1}}{\partial b_t} + (C_{t+1})^{-\sigma}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*}-1\right)\frac{\partial Y_{t+1}}{\partial b_t}$$
$$+ (C_{t+1})^{-\sigma}\frac{Y_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*}-1\right)\frac{\partial \Pi_{t+1}}{\partial b_t} + (C_{t+1})^{-\sigma}\frac{Y_{t+1}}{\Pi^*}\frac{\Pi_{t+1}}{\Pi^*}\frac{\partial \Pi_{t+1}}{\partial b_t}$$

$$= -\sigma(C_{t+1})^{-\sigma-1}Y_{t+1}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*}-1\right)\frac{\partial C_{t+1}}{\partial b_t} + (C_{t+1})^{-\sigma}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*}-1\right)\frac{\partial Y_{t+1}}{\partial b_t}$$
$$+ (C_{t+1})^{-\sigma}\frac{Y_{t+1}}{\Pi^*}\left(\frac{2\Pi_{t+1}}{\Pi^*}-1\right)\frac{\partial \Pi_{t+1}}{\partial b_t}$$

Note we are assuming $E_t\left[Z_b\left(X(s_{t+1})\right)\right] = \partial E_t\left[Z\left(X(s_{t+1})\right)\right]/b_t$, which is normally valid using the Interchange of Integration and Differentiation Theorem. Then the problem is to find a vector-valued function X that Γ maps to the zero function. Projection methods, hence, can be used.

Following the notation convention in the literature, we simply use $s = (b, a)$ to denote the current state of the economy $s_t = (b_{t-1}, a_t)$, and s' to represent next period state that evolves according to the law of motion specified above. The Chebyshev collocation method with time iteration which we use to solve this nonlinear system can be described as follows:

1. Define the collocation nodes and the space of the approximating functions:

   - Choose an order of approximation (i.e., the polynomial degrees) n_b and n_a for each dimension of the state space $s = (b, a)$, then there are $N_s = (n_b + 1) \times (n_a + 1)$ nodes in the state space. Let $S = (S_1, S_2, ..., S_{N_s})$ denote the set of collocation nodes.
   - Compute the $n_b + 1$ and $n_a + 1$ roots of the Chebychev polynomial of order $n_b + 1$ and $n_a + 1$ as
   $$z_b^i = \cos\left(\frac{(2i-1)\pi}{2(n_b+1)}\right), \text{ for } i = 1, 2, ..., n_b + 1.$$
   $$z_a^i = \cos\left(\frac{(2i-1)\pi}{2(n_a+1)}\right), \text{ for } i = 1, 2, ..., n_a + 1.$$
   - Compute collocation points a_i as
   $$a_i = \frac{\overline{a}+\underline{a}}{2} + \frac{\overline{a}-\underline{a}}{2}z_a^i = \frac{\overline{a}-\underline{a}}{2}\left(z_a^i+1\right) + \underline{a}$$
   for $i = 1, 2, ..., n_a + 1$. Note that the number of collocation nodes is $n_a + 1$. Similarly, compute collocation points b_i as
   $$b_i = \frac{\overline{b}+\underline{b}}{2} + \frac{\overline{b}-\underline{b}}{2}z_b^i = \frac{\overline{b}-\underline{b}}{2}\left(z_b^i+1\right) + \underline{b}$$
   for $i = 1, 2, ..., n_b + 1$, which map $[-1, 1]$ into $[\underline{b}, \overline{b}]$. Note that
   $$S = \{(b_i, a_j) \mid i = 1, 2, ..., n_b + 1,\ j = 1, 2, ..., n_a + 1\}$$
   that is, the tensor grids, with $S_1 = (b_1, a_1)$, $S_2 = (b_1, a_2)$, ..., $S_{N_s} = (b_{n_b+1}, a_{n_a+1})$.

- The space of the approximating functions, denoted as Ω, is a matrix of two-dimensional Chebyshev polynomials. More specifically,

$$\Omega(S)=\begin{bmatrix}\Omega(S_1)\\ \Omega(S_2)\\ \vdots\\ \Omega(S_{n_a+1})\\ \vdots\\ \Omega(S_{N_s})\end{bmatrix}=$$

$$=\begin{bmatrix}1 & T_0(\xi(b_1)T_1(\xi(a_1)) & T_0(\xi(b_1)T_2(\xi(a_1)) & \cdots & T_{n_b}(\xi(b_1)T_{n_a}(\xi(a_1))\\ 1 & T_0(\xi(b_1)T_1(\xi(a_2)) & T_0(\xi(b_1)T_2(\xi(a_2)) & \cdots & T_{n_b}(\xi(b_1)T_{n_a}(\xi(a_2))\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 1 & T_0(\xi(b_1)T_1(\xi(a_{n_a+1})) & T_0(\xi(b_1)T_2(\xi(a_{n_a+1})) & \cdots & T_0(\xi(b_1)T_{n_a}(\xi(a_{n_a+1}))\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 1 & T_0(\xi(b_{n_b+1})T_1(\xi(a_{n_a+1})) & T_0(\xi(b_{n_b+1})T_2(\xi(a_{n_a+1})) & \cdots & T_0(\xi(b_{n_b+1})T_{n_a}(\xi(a_{n_a+1}))\end{bmatrix}_{N_s\times N_s}$$

where $\xi(x)=2\left(x-\underline{x}\right)/\left(\overline{x}-\underline{x}\right)-1$ maps the domain of $x\in[\underline{x},\overline{x}]$ into $[-1,1]$.

- Then, at each node $s\in S$, policy functions $X(s)$ are approximated by $X(s)=\Omega(s)\Theta_X$,

where

$$\Theta_X=\left[\theta^c,\theta^y,\theta^\pi,\theta^b,\theta^\tau,\theta^{\widetilde{p}},\theta^g,\theta^{\lambda_1},\theta^{\lambda_2},\theta^{\lambda_3}\right]$$

is a $N_s\times 10$ matrix of the approximating coefficients.

2. Formulate an initial guess for the approximating coefficients, Θ_X^0, and specify the stopping rule ϵ_{tol}, say, 10^{-6}.

3. At each iteration j, we can get an updated Θ_X^j by implement the following time iteration step:

- At each collocation node $s\in S$, compute the possible values of future policy functions $X(s')$ for $k=1,...,q$. That is,

$$X(s')=\Omega(s')\Theta_X^{j-1}$$

where q is the number of Gauss-Hermite quadrature nodes. Note that

$$\Omega(s')=T_{j_b}(\xi(b'))T_{j_a}\left(\xi(a')\right)$$

is a $q\times N_s$ matrix, with $b'=\widehat{b}(s;\theta^b)$, $a'=\rho_a a+z_k\sqrt{2\sigma_a^2}$, $j_b=0,...,n_b$, and $j_a=0,...,n_a$. The hat symbol indicates the corresponding approximate policy functions, so $\widehat{b}$ is the approximate policy for real debt, for example. Similarly, the two auxilliary functions can be calculated as follows:

$$M(s')\approx\left(\widehat{C}(s';\theta^c)\right)^{-\sigma}\widehat{Y}(s';\theta^y)\frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*}\left(\frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*}-1\right)$$

and,

$$L(s')\approx\left(\widehat{C}(s';\theta^c)\right)^{-\sigma}\left(\widehat{\Pi}(s';\theta^\pi)\right)^{-1}\left(1+\frac{\rho\widehat{P^M}\left(s';\theta^{\widetilde{p}}\right)}{\Pi^*-\rho\beta}\right)$$

Note that we use $\widetilde{P}_t^M = (\Pi^* - \rho\beta)\, P_t^M$ rather than P_t^M in numerical analysis, since the former is far less sensitive to maturity structure variations.

- Now calculate the expectation terms $E\left[Z\left(X(s')\right)\right]$ at each node s. Let ω_k denote the weights for the quadrature, then

$$E\left[M(s')\right] \approx \frac{1}{\sqrt{\pi}} \sum_{k=1}^{q} \omega_k \left(\widehat{C}(s';\theta^c)\right)^{-\sigma} \widehat{Y}(s';\theta^y) \frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*} \left(\frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*} - 1\right) \equiv \overline{M}\left(s', q\right)$$

$$E\left[L(s')\right] \approx \frac{1}{\sqrt{\pi}} \sum_{k=1}^{q} \omega_k \left(\widehat{C}(s';\theta^c)\right)^{-\sigma} \left(\widehat{\Pi}(s';\theta^\pi)\right)^{-1} \left(1 + \frac{\rho\widehat{P^M}\left(s';\theta^{\widetilde{p}}\right)}{\Pi^* - \rho\beta}\right) \equiv \overline{L}\left(s', q\right)$$

and

$$E_t\left[\left(\frac{1+\rho P_{t+1}^M}{\Pi_{t+1}}\right)\lambda_{3t+1}\right] \approx \frac{1}{\sqrt{\pi}} \sum_{k=1}^{q} \omega_k \left(\frac{1 + \frac{\rho\widehat{P^M}\left(s';\theta^{\widetilde{p}}\right)}{\Pi^* - \rho\beta}}{\widehat{\Pi}(s';\theta^\pi)}\right) \widehat{\lambda_3}(s';\theta^{\lambda_3}) \equiv \Lambda\left(s', q\right).$$

Hence,

$$E\left[Z\left(X(s')\right)\right] \approx E\left[\widehat{Z}\left(X(s')\right)\right] = \begin{bmatrix} \overline{M}\left(s', q\right) \\ \overline{L}\left(s', q\right) \\ \Lambda\left(s', q\right) \end{bmatrix}$$

- Next calculate the partial derivatives under expectation $E\left[Z_b\left(X(s')\right)\right]$.

- Note that we only need to compute $\partial C_{t+1}/\partial b_t$, $\partial Y_{t+1}/\partial b_t$, $\partial \Pi_{t+1}/\partial b_t$ and $\partial P_{t+1}^M/\partial b_t$, which are given as follows:

$$\frac{\partial C_{t+1}}{\partial b} \approx \sum_{j_b=0}^{n_b} \sum_{j_a=0}^{n_a} \frac{2\theta^c_{j_b j_a}}{\overline{b} - \underline{b}} T'_{j_b}(\xi(b')) T_{j_a}(\xi(a')) \equiv \widehat{C}_b\left(s'\right)$$

$$\frac{\partial Y_{t+1}}{\partial b_t} \approx \sum_{j_b=0}^{n_b} \sum_{j_a=0}^{n_a} \frac{2\theta^y_{j_b j_a}}{\overline{b} - \underline{b}} T'_{j_b}(\xi(b')) T_{j_a}(\xi(a')) \equiv \widehat{Y}_b\left(s'\right)$$

$$\frac{\partial \Pi_{t+1}}{\partial b_t} \approx \sum_{j_b=0}^{n_b} \sum_{j_a=0}^{n_a} \frac{2\theta^\pi_{j_b j_a}}{\overline{b} - \underline{b}} T'_{j_b}(\xi(b')) T_{j_a}(\xi(a')) \equiv \widehat{\Pi}_b\left(s'\right)$$

$$\frac{\partial P_{t+1}^M}{\partial b_t} \approx \sum_{j_b=0}^{n_b} \sum_{j_a=0}^{n_a} \frac{2\theta^{\widetilde{p}}_{j_b j_a}}{\left(\overline{b} - \underline{b}\right)\left(\Pi^* - \rho\beta\right)} T'_{j_b}(\xi(b_i)) T_{j_a}(\xi(a_j)) \equiv \widehat{P}_b^M\left(s'\right)$$

Hence, we can approximate the two partial derivatives under expectation

$$\frac{\partial E\left[M(s')\right]}{\partial b}$$

$$\approx \frac{1}{\sqrt{\pi}} \sum_{k=1}^{q} \omega_k \begin{bmatrix} -\sigma\left(\widehat{C}(s';\theta^c)\right)^{-\sigma-1} \widehat{Y}(s';\theta^y) \frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*} \left(\frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*} - 1\right) \widehat{C}_b\left(s'\right) \\ + \left(\widehat{C}(s';\theta^c)\right)^{-\sigma} \frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*} \left(\frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*} - 1\right) \widehat{Y}_b\left(s'\right) \\ + \left(\widehat{C}(s';\theta^c)\right)^{-\sigma} \frac{\widehat{\Pi}(s';\theta^\pi)}{\Pi^*} \left(\frac{2\widehat{\Pi}(s';\theta^\pi)}{\Pi^*} - 1\right) \widehat{\Pi}_b\left(s'\right) \end{bmatrix}$$

$$\equiv \widehat{M}_b\left(s', q\right),$$

$$\frac{\partial E\left[L(s')\right]}{\partial b}$$

$$\approx \frac{1}{\sqrt{\pi}} \sum_{k=1}^{q} \omega_k \left[\begin{array}{c} -\sigma \left(\widehat{C}(s';\theta^c)\right)^{-\sigma-1} \left(\widehat{\Pi}(s';\theta^\pi)\right)^{-1} (1 + \frac{\rho \widehat{P^M}\left(s';\theta^{\widetilde{p}}\right)}{\Pi^* - \rho\beta}) \widehat{C}_b\left(s'\right) \\ - \left(\widehat{C}(s';\theta^c)\right)^{-\sigma} \left(\widehat{\Pi}(s';\theta^\pi)\right)^{-2} (1 + \frac{\rho \widehat{P^M}\left(s';\theta^{\widetilde{p}}\right)}{\Pi^* - \rho\beta}) \widehat{\Pi}_b\left(s'\right) \\ +\rho \left(\widehat{C}(s';\theta^c)\right)^{-\sigma} \left(\widehat{\Pi}(s';\theta^\pi)\right)^{-1} \widehat{P}_b^M\left(s'\right) \end{array} \right]$$

$$\equiv \widehat{L}_b\left(s', q\right).$$

That is,

$$E\left[Z_b\left(X(s')\right)\right] \approx E\left[\widehat{Z}_b\left(X(s')\right)\right] = \left[\begin{array}{c} \widehat{M}_b\left(s', q\right) \\ \widehat{L}_b\left(s', q\right) \end{array} \right]$$

4. At each collocation node s, solve for $X(s)$ such that

$$\Gamma\left(s, X(s), E\left[\widehat{Z}\left(X(s')\right)\right], E\left[\widehat{Z}_b\left(X(s')\right)\right]\right) = 0$$

   The equation solver *csolve* written by Christopher A. Sims is employed to solve the resulted system of nonlinear equations. With $X(s)$ at hand, we can get the corresponding coeffcient

$$\widehat{\Theta}_X^j = \left(\Omega\left(S\right)^T \Omega\left(S\right)\right)^{-1} \Omega\left(S\right)^T X(s)$$

5. Update the approximating coefficients, $\Theta_X^j = \eta \widehat{\Theta}_X^j + (1-\eta)\,\Theta_X^{j-1}$, where $0 \le \eta \le 1$ is some dampening parameter used for improving convergence.

6. Check the stopping rules. If $\left\|\Theta_X^j - \Theta_X^{j-1}\right\| < \epsilon_{tol}$, then stop, else update the approximation coefficients and go back to step 3.

When implementing the above algorithm, we start from lower order Chebyshev polynomials and some reasonable initial guess. Then, we increase the order of approximation and take as starting value the solution from the previous lower order approximation. This informal homotopy continuation idea ensures us to find a solution.

Remark. Given the fact that the price P_t^M fluctuates significantly for larger ρ, in numerical analysis, we scale rule for P_t^M by $(\Pi^* - \rho\beta)$, that is, $\widetilde{P}_t^M = (\Pi^* - \rho\beta)P_t^M$. In this way, the steady state of $\widetilde{P}_t^M$ is very close to β, and $\widetilde{P}_t^M$ does not differ hugely as we change the maturity structure.

C.3 Optimal Policy Under Discretion With Endogenous Short-Term Debt

In this case, the government is allowed to issue new bonds of a different maturity and swap these for existing bonds, in a way which does not affect the wealth of the bond holders at the time of the swap, such that the exchange is voluntary.

The policy under discretion in this case can be described as a set of decision rules for $\{C_t, Y_t, \Pi_t, b_t, \tau_t, G_t, b_t^S\}$ which maximise,

$$V(b_{t-1}, A_t, b_{t-1}^S) = \max\left\{\frac{C_t^{1-\sigma}}{1-\sigma} + \chi\frac{G_t^{1-\sigma_g}}{1-\sigma_g} - \frac{(Y_t/A_t)^{1+\varphi}}{1+\varphi} + \beta E_t\left[V(b_t, A_{t+1}, b_t^S)\right]\right\}$$

subject to the following constraints:

$$Y_t\left[1 - \frac{\phi}{2}\left(\frac{\Pi_t}{\Pi^*} - 1\right)^2\right] = C_t + G_t$$

$$\begin{aligned} 0 &= (1-\epsilon) + \epsilon mc_t - \phi\frac{\Pi_t}{\Pi^*}\left(\frac{\Pi_t}{\Pi^*} - 1\right) \\ &+ \phi\beta E_t\left[\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\frac{Y_{t+1}}{Y_t}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*} - 1\right)\right] \end{aligned}$$

$$\begin{aligned} &\beta E_t\left\{\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\left(\frac{P_t}{P_{t+1}}\right)(1+\rho P_{t+1}^M)\right\} b_t + \beta E_t\left\{\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\left(\frac{P_t}{P_{t+1}}\right)\right\} b_t^S \\ &= \left(1 + \rho\beta E_t\left\{\left(\frac{C_t}{C_{t+1}}\right)^{\sigma}\left(\frac{P_t}{P_{t+1}}\right)(1+\rho P_{t+1}^M)\right\}\right)\frac{b_{t-1}}{\Pi_t} + \frac{b_{t-1}^S}{\Pi_t} \\ &- \left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{Y_t}{A_t}\right)^{1+\varphi} C_t^{\sigma} + G_t \end{aligned}$$

where b_t^S is the level of real short-term debt.

Defining auxilliary functions,

$$M(b_t, A_{t+1}, b_t^S) = (C_{t+1})^{-\sigma}\, Y_{t+1}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*} - 1\right)$$

$$L(b_t, A_{t+1}, b_t^S) = (C_{t+1})^{-\sigma}(\Pi_{t+1})^{-1}(1+\rho P_{t+1}^M)$$
$$K\left(b_t, A_{t+1}, b_t^S\right) = C_{t+1}^{-\sigma}\Pi_{t+1}^{-1}$$

we can rewrite the NKPC and government budget constraints as, respectively,

$$(1-\epsilon) + \epsilon(1-\tau_t)^{-1}Y_t^{\varphi}C_t^{\sigma}A_t^{-1-\varphi} - \phi\frac{\Pi_t}{\Pi^*}\left(\frac{\Pi_t}{\Pi^*} - 1\right) + \phi\beta C_t^{\sigma}Y_t^{-1}E_t\left[M(b_t, A_{t+1}, b_t^S)\right] = 0$$

$$\begin{aligned} 0 &= \beta b_t C_t^{\sigma} E_t\left[L(b_t, A_{t+1}, b_t^S)\right] + \beta b_t^S C_t^{\sigma} E_t\left[K\left(b_t, A_{t+1}, b_t^S\right)\right] - \frac{b_{t-1}}{\Pi_t}\left(1 + \rho\beta C_t^{\sigma}E_t\left[L(b_t, A_{t+1}, b_t^S)\right]\right) \\ &- \frac{b_{t-1}^S}{\Pi_t} + \left(\frac{\tau_t}{1-\tau_t}\right)\left(\frac{Y_t}{A_t}\right)^{1+\varphi} C_t^{\sigma} - G_t \end{aligned}$$

The Lagrangian for the policy problem can be written as,

$$\mathcal{L} = \left\{ \frac{C_t^{1-\sigma}}{1-\sigma} + \chi \frac{G_t^{1-\sigma_g}}{1-\sigma_g} - \frac{(Y_t/A_t)^{1+\varphi}}{1+\varphi} + \beta E_t[V(b_t, A_{t+1}, b_t^S)] \right\}$$

$$+ \lambda_{1t} \left[Y_t \left(1 - \frac{\phi}{2} \left(\frac{\Pi_t}{\Pi^*} - 1 \right)^2 \right) - C_t - G_t \right]$$

$$+ \lambda_{2t} \left[\begin{array}{c} (1-\epsilon) + \epsilon(1-\tau_t)^{-1} Y_t^{\varphi} C_t^{\sigma} A_t^{-1-\varphi} - \phi \frac{\Pi_t}{\Pi^*} \left(\frac{\Pi_t}{\Pi^*} - 1 \right) \\ + \phi \beta C_t^{\sigma} Y_t^{-1} E_t \left[M(b_t, A_{t+1}, b_t^S) \right] \end{array} \right]$$

$$+ \lambda_{3t} \left[\begin{array}{c} \beta b_t C_t^{\sigma} E_t \left[L(b_t, A_{t+1}, b_t^S) \right] + \beta b_t^S C_t^{\sigma} E_t \left[K \left(b_t, A_{t+1}, b_t^S \right) \right] \\ - \frac{b_{t-1}}{\Pi_t} \left(1 + \rho \beta C_t^{\sigma} E_t \left[L(b_t, A_{t+1}, b_t^S) \right] \right) \\ - \frac{b_{t-1}^S}{\Pi_t} + \left(\frac{\tau_t}{1-\tau_t} \right) \left(\frac{Y_t}{A_t} \right)^{1+\varphi} C_t^{\sigma} - G_t \end{array} \right]$$

We can write the first order conditions for the policy problem as follows:

Consumption,

$$C_t^{-\sigma} - \lambda_{1t} + \lambda_{2t} \left[\sigma \epsilon (1-\tau_t)^{-1} Y_t^{\varphi} C_t^{\sigma-1} A_t^{-1-\varphi} + \sigma \phi \beta C_t^{\sigma-1} Y_t^{-1} E_t \left[M(b_t, A_{t+1}, b_t^S) \right] \right]$$

$$+ \lambda_{3t} \left[\begin{array}{c} \sigma \beta b_t C_t^{\sigma-1} E_t \left[L(b_t, A_{t+1}, b_t^S) \right] + \sigma \beta b_t^S C_t^{\sigma-1} E_t \left[K \left(b_t, A_{t+1}, b_t^S \right) \right] \\ - \rho \sigma \beta \frac{b_{t-1}}{\Pi_t} C_t^{\sigma-1} E_t \left[L(b_t, A_{t+1}, b_t^S) \right] + \sigma \left(\frac{\tau_t}{1-\tau_t} \right) \left(\frac{Y_t}{A_t} \right)^{1+\varphi} C_t^{\sigma-1} \end{array} \right] = 0$$

Government spending,

$$\chi G_t^{-\sigma_g} - \lambda_{1t} - \lambda_{3t} = 0$$

Output,

$$-Y_t^{\varphi} A_t^{-1-\varphi} + \lambda_{1t} \left[1 - \frac{\phi}{2} \left(\frac{\Pi_t}{\Pi^*} - 1 \right)^2 \right]$$

$$+ \lambda_{2t} \left[\epsilon \varphi (1-\tau_t)^{-1} Y_t^{\varphi-1} C_t^{\sigma} A_t^{-1-\varphi} - \phi \beta C_t^{\sigma} Y_t^{-2} E_t \left[M(b_t, A_{t+1}, b_t^S) \right] \right]$$

$$+ \lambda_{3t} \left[(1+\varphi) Y_t^{\varphi} C_t^{\sigma} \left(\frac{\tau_t}{1-\tau_t} \right) A_t^{-1-\varphi} \right] = 0$$

Taxation,

$$\epsilon \lambda_{2t} + \lambda_{3t} Y_t = 0$$

Inflation,

$$-\lambda_{1t} \left[Y_t \frac{\phi}{\Pi^*} \left(\frac{\Pi_t}{\Pi^*} - 1 \right) \right] - \lambda_{2t} \left[\frac{\phi}{\Pi^*} \left(\frac{2\Pi_t}{\Pi^*} - 1 \right) \right]$$

$$+ \lambda_{3t} \left[\frac{b_{t-1}}{\Pi_t^2} \left(1 + \rho \beta C_t^{\sigma} E_t \left[L(b_t, A_{t+1}, b_t^S) \right] \right) + \frac{b_{t-1}^S}{\Pi_t^2} \right] = 0$$

Government debt, b_t,

$$\beta E_t[V_1(b_t, A_{t+1}, b_t^S)] + \lambda_{2t} \left[\phi \beta C_t^{\sigma} Y_t^{-1} E_t \left[M_1(b_t, A_{t+1}, b_t^S) \right] \right]$$

$$+ \beta C_t^{\sigma} \lambda_{3t} \left[\begin{array}{c} E_t \left[L(b_t, A_{t+1}, b_t^S) \right] + b_t E_t \left[L_1(b_t, A_{t+1}, b_t^S) \right] + b_t^S E_t \left[K_1 \left(b_t, A_{t+1}, b_t^S \right) \right] \\ - \rho \frac{b_{t-1}}{\Pi_t} E_t \left[L_1(b_t, A_{t+1}, b_t^S) \right] \end{array} \right] = 0$$

where

$$V_1(b_t, A_{t+1}, b_t^S) \equiv \partial V(b_t, A_{t+1}, b_t^S)/\partial b_t$$
$$L_1(b_t, A_{t+1}, b_t^S) \equiv \partial L(b_t, A_{t+1}, b_t^S)/\partial b_t$$
$$M_1(b_t, A_{t+1}, b_t^S) \equiv \partial M(b_t, A_{t+1}, b_t^S)/\partial b_t$$
$$K_1(b_t, A_{t+1}, b_t^S) \equiv \partial K(b_t, A_{t+1}, b_t^S)/\partial b_t$$

Short-term government debt, b_t^S,

$$\beta E_t[V_3(b_t, A_{t+1}, b_t^S)] + \lambda_{2t}\left[\phi\beta C_t^\sigma Y_t^{-1} E_t\left[M_3(b_t, A_{t+1}, b_t^S)\right]\right]$$

$$+\beta C_t^\sigma \lambda_{3t}\left[\begin{array}{c} b_t E_t\left[L_3(b_t, A_{t+1}, b_t^S)\right] + E_t\left[K\left(b_t, A_{t+1}, b_t^S\right)\right] + b_t^S E_t\left[K_3\left(b_t, A_{t+1}, b_t^S\right)\right] \\ -\rho\frac{b_{t-1}}{\Pi_t} E_t\left[L_3(b_t, A_{t+1}, b_t^S)\right] \end{array}\right] = 0$$

where

$$V_3(b_t, A_{t+1}, b_t^S) \equiv \partial V(b_t, A_{t+1}, b_t^S)/\partial b_t^S$$
$$L_3(b_t, A_{t+1}, b_t^S) \equiv \partial L(b_t, A_{t+1}, b_t^S)/\partial b_t^S$$
$$M_3(b_t, A_{t+1}, b_t^S) \equiv \partial M(b_t, A_{t+1}, b_t^S)/\partial b_t^S$$
$$K_3(b_t, A_{t+1}, b_t^S) \equiv \partial K(b_t, A_{t+1}, b_t^S)/\partial b_t^S$$

Note that by the envelope theorem,

$$V_1(b_{t-1}, A_t, b_{t-1}^S) = -\frac{\lambda_{3t}}{\Pi_t}\left(1 + \rho\beta C_t^\sigma E_t\left[L(b_t, A_{t+1}, b_t^S)\right]\right)$$

$$= -\frac{\lambda_{3t}}{\Pi_t}\left(1 + \rho P_t^M\right)$$

$$V_3(b_{t-1}, A_t, b_t^S) = -\frac{\lambda_{3t}}{\Pi_t}$$

hence,

$$V_1(b_t, A_{t+1}, b_t^S) = \frac{\lambda_{3t+1}}{\Pi_{t+1}}\left(1 + \rho P_{t+1}^M\right)$$

$$V_3(b_t, A_{t+1}, b_t^S) = -\frac{\lambda_{3t+1}}{\Pi_{t+1}}$$

and the FOCs for government debt b_t and b_t^S can be rewritten as, respectively,

$$-\beta E_t\left[\frac{\lambda_{3t+1}}{\Pi_{t+1}}(1 + \rho P_{t+1}^M)\right] + \lambda_{2t}\phi\beta C_t^\sigma Y_t^{-1} E_t\left[M_1(b_t, A_{t+1}, b_t^S)\right]$$

$$+\beta C_t^\sigma \lambda_{3t}\left[\begin{array}{c} E_t\left[L(b_t, A_{t+1}, b_t^S)\right] + b_t E_t\left[L_1(b_t, A_{t+1}, b_t^S)\right] + b_t^S E_t\left[K_1\left(b_t, A_{t+1}, b_t^S\right)\right] \\ -\rho\frac{b_{t-1}}{\Pi_t} E_t\left[L_1(b_t, A_{t+1}, b_t^S)\right] \end{array}\right] = 0$$

and

$$-\beta E_t[\frac{\lambda_{3t+1}}{\Pi_{t+1}}] + \lambda_{2t}\phi\beta C_t^\sigma Y_t^{-1} E_t\left[M_3(b_t, A_{t+1}, b_t^S)\right]$$

$$+\beta C_t^{\sigma}\lambda_{3t}\left[\begin{array}{c} b_t E_t\left[L_3(b_t,A_{t+1},b_t^S)\right]+E_t\left[K\left(b_t,A_{t+1},b_t^S\right)\right]+b_t^S E_t\left[K_3\left(b_t,A_{t+1},b_t^S\right)\right] \\ -\rho\frac{b_{t-1}}{\Pi_t}E_t\left[L_3(b_t,A_{t+1},b_t^S)\right] \end{array}\right]=0$$