DISPUTES, DEBT AND EQUITY *

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We show how the prospect of disputes over firms' revenue reports promotes debt financing over equity. These findings are presented within a costly state verification model with a risk averse entrepreneur. The prospect of disputes encourages incentive regimes which limit penalties and avoid stochastic monitoring, even when the lender can commit to stochastic enforcement strategies. Consequently, optimal contracts shift away from equity and toward standard debt. For a useful special case of the model, closed form solutions are presented for leverage and consumption allocations under efficient debt contracts.

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INTRODUCTION AND OVERVIEW

What form should an optimal contract take to handle the prospect of wrongful penalties? Our answer is that standard debt contracts are often optimal. In other words, a theory of debt is proposed based on inaccurate auditing in a costly state verification framework.

As we detail presently, costly state verification environments imply that equity contracts are optimal. However, that conclusion is shown to rest crucially on the efficacy of the audit technology. We introduce wrongful penalties through an imperfect audit technology.[1] With *perfect* auditing, optimal contracts can economize on audit costs ex post by committing to sufficiently severe penalties ex ante. Consequently, deterministic auditing is not optimal. Moreover, enjoying access to a perfect audit technology, agents may write contracts which pass on small fluctuations in revenue to security holders. However, a key insight under *imperfect* auditing is that the prospect of disputes and wrongful penalties restricts acceptable contracts to using smaller penalties than would otherwise be the case.[2] Smaller penalties come at the cost of encouraging fraudulent reports for any given repayment schedule. So, whilst increasing the frequency of audits can help with risk sharing, it also implies levying more wrongful penalties.

It turns out that the marginal gain to auditing low reports is positive even as the probablility of audits approaches unity.[3] To see why auditing is often efficient despite its cost and inaccuracy, note that borrowers who suffer poor returns ('low types') and who are not audited will have lower consumption allocations than otherwise as they are indistinguishable from

[1] Our results carry over to other situations where the lender or bankruptcy court might erroneously dispute the borrower's revenue report.

[2] Our model is static, and our penalties are just units of the consumption good, but this reasoning carries over to alternative settings where alternative enforcement schemes such as non-pecuniary penalties or exclusion may be applied. When disputes are possible, even honest borrowers prefer the penalties for dishonesty to be smaller, all else equal.

[3] Note that an exogenous restriction to penalties does not have the same effect as audit errors. When there are audit errors, agents always prefer penalties to be lower, all else equal. When penalties are exogenously limited, there is no desire for them to be lower than the exogenous limit. As long as exogenous limits to penalties are not trivial, stochastic audit regimes are optimal under perfect audits.

high types. And whilst their consumption is not as low as the entrepreneurs who suffer errors (i.e., Type-I errors), it is still somewhat lower than low types whose reports are verified. Hence, whilst fewer entrepreneurs suffer errors when audits are stochastic, fewer also have their reports verified. That benefit from auditing outweighs the cost.[4]

On the other hand, attempting to pass on small fluctuations in revenue would require a high probability of audit but would entail the possibility of an error or dispute. At some point, it might no longer be worthwhile to attempt to pass on these risks: It may be better just to pay the principal plus interest.

The upshot is that we find that the optimal external finance contract often combines deterministic audits following low revenue reports with no audits for revenue reports above a certain cutoff. Following most reports, the borrower's repayment is independent of marginal differences in revenue: the borrower simply repays the principal plus interest. Moreover, when errors are rare, the optimal repayment following an overturned income report is just equal to the contracted coupon plus principal of the loan. Such features resemble standard debt contracts.[5,6] The key results are contained largely in Theorem 2 and Proposition 3 below.

[4]There is, however, an additional benefit from stochastic audits since the higher leverage that accompanies it implies high types enjoy higher consumption. In general, as we show below, that consideration does not outweigh the benefits from an always-audit contract.

[5]Earlier contributions to CSV problems with audit errors have focused on insurance problems in the context of a risky endowment. Haubrich (1995) shows that weakly informative audits are rarely used in efficient contracts. Alary and Gollier (2004) study an example with no commitment to audits, showing that the occurence of strategic default is dependent on the preferences of the agent. Imperfect signals are also commonly employed in the law enforcement literature. See Polinsky and Shavell (2007) for an excellent summary.

[6]A different literature assumes that project outcomes are observable, yet entrepreneurial actions are partially observable. Efficient contracts must encourage entrepreneurs to exert privately costly effort. In these models, the concepts of debt and equity finance are related solely to the optimal sensitivity of repayments to project outcomes. A recent example which rationalises a combination of debt and equity in this setting with partially observable actions and limited enforceability is Ellingsen and Kristiansen (2011).

0.1 DETERMINISTIC INCENTIVE REGIMES AND LEVERAGE

Introducing imperfect audits encourages both deterministic audit regimes, when risk sharing is considered to be of high value, and also the complete removal of audits, when risk sharing is considered to be of low value. The interaction between leverage and costly, imperfect auditing underpins the finding that deterministic incentive schemes are generally optimal. Note that leverage and audit probability are similar in that higher leverage increases expected consumption and the spread of consumption outturns; so too does a decrease in audit probabilities. So, for low levels of borrowing and hence low levels of risk, audits are less desirable. However, if borrowing is very high there will be a large impact on consumption if a low return is mistaken for a high return, what we call a type-I error. That implies that there is an *endogenous borrowing limit* and that the audit probability goes to zero as borrowing approaches that limit. For intermediate levels of borrowing equilibrium auditing is typically deterministic; that is, of probability one.

That non-monotonic relationship between audit probabilities and leverage is perhaps surprising. However, more surprising is what we label a *bang-bang* result: Efficient contracts can jump from being non-contingent to standard debt contracts with deterministic auditing in low states, in response to marginal increases in project risk. Moreover, there is a discontinuous decrease in optimal leverage, an increase in default, an increase in expected monitoring costs and a drop in average consumption. We are able to characterize analytically the trade-offs that occur at the point when optimal contracts change in that way. At that point, there are two contracts which deliver the same level of utility; one a high-leverage/never-audit contract, the other a low-leverage/standard debt contract.

Audit costs also play an important role in determining optimal leverage. When audit costs are low, optimal leverage is such as to permit large gains from insurance or auditing. In that case, 'extreme' incentive regimes tend to be optimal and auditing strategies are, again, deterministic; specifically, always audit contracts are optimal.[7]

Interior solutions for audit probability are possible but these are somewhat unusual in our set-up because they require very high audit costs in addition to high project risk.

1 Literature

Equity finance typically allows issuers to reduce repayments or dividends in bad times whilst reductions in the value of assets are shared between borrowers and lenders. Debt finance is more rigid. Debts are only reduced or discharged in bankruptcy, which follows large falls in income or asset values. So, surely it would be better if there was less debt and more equity?

Townsend (1979) was first to propose an explanation for the prevalence of debt contracts. He shows that when a risk averse borrower's income is costly to verify a standard debt contract is superior either to a strict debt contract, where repayments are constant across states, or a standard equity contract, where repayments are proportional to the borrower's income. The difficulty with the equity contract is that to ensure the borrower does not misreport income the investor needs to undertake a costly audit regardless of the report. A superior contract prescribes audits and risk sharing only following sufficiently low reports, when the borrower's marginal utility and sensitivity to risk are highest. If the borrower's income is sufficiently high, they make a fixed repayment and absorb any remaining income risk at the margin. Such a contract is the standard debt contract that is widespread in personal and corporate loan markets.

Townsend's analysis constrained agents to deterministic auditing regimes. However, he suggested a better contract might employ a stochastic auditing schedule. Perhaps following a very low report an audit would be highly

[7] Gale and Hellwig (1985) also study the effects of audit costs and risk aversion on leverage in a costly state verification model with perfect and deterministic audits. Our analysis permits stochastic audit regimes, and finds alternative interactions between leverage and the contracting environment: leverage has a dramatic impact on the nature of the efficient contract in our model, and it is the joint determination of leverage and incentive regime which encourages debt contracts in our framework.

likely, and following a high report less so. Using stochastic auditing schemes would allow more risk sharing across states with fewer resources spent on audits across a portfolio of loans. Border and Sobel (1987) and Mookherjee and Png (1989) confirm Townsend's conjecture. In fact, they show that deterministic audit strategies are *never* constrained efficient: Audit strategies should be stochastic, and the probability of audit should be positive even following relatively high revenue reports. Such a contract looks more like equity in the sense that income need not be low before the contract specifies risk sharing.

That risk sharing comes at a cost. A cost that is not captured in the benchmark model. In order to ensure truth-telling when the probability of audit is low, audits that contradict the borrower's report can result in penalties far larger than the amount borrowed. If that audit technology were to contradict a *truthful* report, then the prospect of sizeable, wrongful penalties might render such contracts unacceptable to the borrower. Indeed, even if the entrepreneur were merely to *fear* that audits may not be perfect, or that their truthful report may be disputed by the lender or bankruptcy court, they would likely baulk at a contract that leaves open the prospect of large penalties following disputed reports. In short, equity-like contracts provide more insurance across states, but may exacerbate already bad situations for a borrower. Hence the motivation of this paper.

1.1 COMMITMENT

This paper, along with the aforementioned studies, considers an environment where the lender is able to commit ex ante to an incentive regime which is wasteful ex post. That commitment may indicate a concern for reputation, or delegation to a specialised auditor or bankruptcy court as in Melumad and Mookherjee (1989). Krasa and Villamil (2000) investigate what happens when lenders cannot commit to costly audits. That lack of commitment means the revelation principle does not hold and in equilibrium borrowers misreport their income with positive probability. It turns out that lack of commitment means that determinstic audits may be a feature of the optimal contract. Audits can only occur if the expected value of penalties levied following audits exceeds the audit costs. If true for a particular reported income, then this report will be audited with certainty. In short, for Krasa and Villamil (2000) the ability to commit implies equity-like contracts are preferable, whereas for us it does not.

The rest of the paper is set out as follows. Section 1 lays out the model environment and the nature of the auditing technology. Section 2 characterizes some key features of efficient contracts. In section 3 we present the perfect audits benchmark. Section 4 explores the imperfect audits case, and contains the key contributions of the paper. Section 5 presents comparative statics for a special case of the model where closed form solutions can be obtained. Section 6 provides a numerical example of a four state version of the model. Efficient contracts under perfect and imperfect audits are compared. Section 7 offers concluding remarks. Appendices contain formal arguments and proofs. Figures are contained in Appendix E.

2 THE ENVIRONMENT

We study the one period problem of a risk averse and credit constrained entrepreneur. The entrepreneur has access to a special technology offering high returns which are uncorrelated with other projects undertaken in the economy.

The outcome of the project is initially private information to the entrepreneur, limiting the sharing of risk between the entrepreneur and their financier (the financial intermediary). Contract repayments are enforceable, but can only be conditioned on public information. The public information available to condition contracts includes any message sent by the entrepreneur, and any audit signal produced by the audit technology.

The entrepreneur makes a take-it-or-leave it contract offer to the financial intermediary, who is well-diversified and perfectly competitive. An efficient contract maximises the entrepreneur's expected utility.

2.1 THE ENTREPRENEUR

The entrepreneur enjoys consumption at the end of the period according to $U(x)$, where $U', -U'' > 0$, and $U'(0) = \infty$. The entrepreneur brings wealth α of the consumption good into the period. Combining the entrepreneur's wealth α with the net funds borrowed from the financial intermediary b, the project produces the consumption good according to stochastic gross return $(\alpha + b)\theta$. In this section, we restrict revenue to be drawn from one of two states, $\theta \in \Theta$, $\Theta = \{\bar{\theta}, \underline{\theta}\}$ and $\bar{\theta} > \underline{\theta}$. This restriction assists the intuition behind our key results, but is not essential for them. Section 6 extends the model to a four-state version, exploring optimal risk sharing across states in the perfect and imperfect audit models.

Following the realisation of their project, the entrepreneur can send a public signal indicating the state and subsequent revenues of the project. Messages m are drawn from $M = \{\bar{m}, \underline{m}\}$, where a message of $\bar{m}$ corresponds to reporting that the entrepreneur has received a high type shock $\bar{\theta}$, and a report of $\underline{m}$ implies a low type shock, $\underline{\theta}$.

2.2 THE FINANCIAL INTERMEDIARY

There exists a well-diversified financial intermediary who can make credible commitments to future actions.[8] Any contract involving the entrepreneur and the financial intermediary is small from the perspective of the financial intermediary's balance sheet. Further, the entrepreneur's return shock θ is uncorrelated with other shocks in the economy, and the returns of other assets/liabilities of the financial intermediary's balance sheet. It follows that the financial intermediary is risk neutral with respect to claims contingent on the entrepreneur's return shock θ.

The financial intermediary operates in a perfectly competitive market. Their opportunity cost of funds is given by ρ, and any contract offering an expected return on possibly state contingent loans exceeding ρ is acceptable

[8]Efficient contracts will require commitment on behalf of the financial intermediary. One might think of this as sustained either through the intermediary's concern for its reputation, or through delegation to a specialist bailiff or auditor as in Melumad and Mookherjee (1989).

to the financial intermediary. This condition is formalised in Definition 5. The opportunity cost of funds could be thought of as some combination of the interest rate paid by a risk free bond, the interest rate paid by the intermediary to their deposit holders, and the intermediary's administrative costs.

The following two assumptions ensure that there are available positive (but finite) gains from trade between the entrepreneur and financial intermediary.

Assumption 1 *Expected project returns exceed the financial intermediary's opportunity cost of funds,* $\sum_{\theta \in \Theta} \pi(\theta)\theta > \rho$.

Assumption 2 *In the low state, project returns are lower than the financial intermediary's opportunity cost of funds,* $\underline{\theta} < \rho$.

Assumption 1 ensures that there are economic gains from diverting resources to the entrepreneur's project, even when the entrepreneur has access to a deposit facility at the bank yielding a risk free return equal to the bank's opportunity cost of funds, ρ. Assumption 1 is strong enough to ensure that $b > -\alpha$.

Assumption 2 specifies that the entrepreneurs' projects are risky. In bad states, a project will yield lower returns than the risk free asset. Assumption 2 will be sufficient to ensure that leverage is finite under efficient contracts when type-I audit errors are present, a result shown in Proposition 3.

2.3 AUDITS

There exists an audit technology which produces a signal $\sigma \in \Sigma$ providing information about the outcome of the entrepreneur's project ex post. The action to undertake an audit is common knowledge, and so is the signal provided, σ. In other words, the entrepreneur knows if (s)he has been audited, and the result of the audit. It is assumed that an audit strategy, contingent on the entrepreneur's reports, can be agreed and committed to ex ante.

The audit technology cost is linear in assets, $\kappa(\alpha + b)$, where κ is a fixed parameter. The signal produced by the audit technology maps from the space of realised shocks θ as follows: If there is no audit, the audit signal is the empty set, $\sigma = \emptyset$. If there is an audit and the true state is $\bar{\theta}$, the audit technology reports $\sigma(\bar{\theta}) = \bar{\sigma}$ with probability $(1 - \eta(\bar{\theta}))$, and $\sigma(\bar{\theta}) = \underline{\sigma}$ with probability $\eta(\bar{\theta})$. If there is an audit and the true state is $\underline{\theta}$, the audit technology reports $\sigma(\underline{\theta}) = \underline{\sigma}$ with probability $(1-\eta(\underline{\theta}))$, and $\sigma(\underline{\theta}) = \bar{\sigma}$ with probability $\eta(\underline{\theta})$.

Assumption 3 *Audits are informative:* $\eta(\bar{\theta}) + \eta(\underline{\theta}) < 1$.

Definition 1 *Conditional upon an audit, a* Type-I error *occurs when the audit technology signals a high type return when the true return is low,* $\sigma(\underline{\theta}) = \bar{\sigma}$. *A* Type-II error *occurs when the audit technology signals a low type return when the true return is high,* $\sigma(\bar{\theta}) = \underline{\sigma}$.

Audit strategies are defined in contracts, and implemented ex post by the financial intermediary. An audit strategy specifies the probability of audit, conditional upon the message sent by the entrepreneur, $q(m)$.

3 CONTRACTS

Definition 2 *A* contract *is an ordered set* $\Gamma = (b, q(m), z(m, \sigma), x(m, \sigma, \theta))$ *where* $b, q(m)$ *are publicly observed actions;* $z(m, \sigma) : \Theta \times \Sigma \to \mathbb{R}$ *is a function mapping publicly observed information to the financial intermediary's ex post receipt from the entrepreneur; and the entrepreneur's consumption allocations are specified by* $x(m, \sigma, \theta) : M \times \Sigma \times \Theta \to \mathbb{R}^+$.

A key motivation for this paper is the search for environments where debt contracts are efficient.

Definition 3 *We specify the following two benchmark contracts.*

a. A non-contingent debt *contract is a contract with constant repayments across all states and messages* $z(m_i, \sigma_j) = z(m_k, \sigma_l)\ \forall m_i, m_k \in \mathcal{M},\ \sigma_j, \sigma_l \in \Sigma$. *Any available audit signals are ignored, and therefore no audits are conducted (*$q(\underline{m}) = 0$*).*

b. A standard debt *contract specifies a constant repayment when either the entrepreneur's message or the audit signal is high, and a lower repayment following a verified low report ($z(\bar{m}, \emptyset) = z(\underline{m}, \bar{\sigma}) > z(\underline{m}, \underline{\sigma})$). All low reports are audited ($q(\underline{m}) = 1$).*

Note that debt contracts in our model do not restrict the entrepreneur borrower to zero consumption following default. In fact, in the examples that we consider, entrepreneurs will enjoy strictly positive consumption in all circumstances, even following a default. This positive consumption could represent income already paid to the entrepreneur during the life of the project, or rights to future earned income after the discharging of debts in bankruptcy.

Budget Constraints *State contingent budget constraints are specified as follows:*

$$(\alpha + b)\theta = z(m, \sigma) + x(m, \sigma, \theta) \qquad \forall (m, \sigma, \theta) \in M \times \Sigma \times \Theta. \tag{1}$$

The left hand side is the revenue received by the entrepreneur from their project, denominated in the consumption good. Following the repayment z, the remainder available for the entrepreneur to consume is x.

Definition 4 *A contract is* incentive compatible *if and only if $m^*(\underline{\theta}) = \underline{m}, m^*(\bar{\theta}) = \bar{m}$ solves the following problem:*

$$m^*(\theta) \in \arg\max_{m(\theta)} (1 - q(m(\theta)))U(x(m(\theta), \emptyset, \theta)) + q(m(\theta))[1 - \eta(\theta)]U(x(m(\theta), \sigma = \theta, \theta))$$
$$+ q(m(\theta))\eta(\theta)U(x(m(\theta), \sigma \neq \theta, \theta)), \qquad \theta \in \{\underline{\theta}, \bar{\theta}\} \tag{2}$$

The consumption allocations on the right hand side of equation 2 are bundles enjoyed by misreporting agents. We can re-write the incentive compatibility constraint with respect to bundles consumed in truth-telling contracts by substituting in the budget constraints (1):

$$m^*(\theta) \in \arg\max_{m(\theta)} (1 - q(m(\theta)))U(x(m(\theta), \emptyset, \theta))$$
$$+ q(m(\theta))[1 - \eta(\theta)]U[x(m(\theta), \sigma = \theta, m(\theta)) + (\alpha + b)(\theta - m(\theta))]$$
$$+ q(m(\theta))\eta(\theta)U[x(m(\theta), \sigma \neq \theta, m(\theta)) + (\alpha + b)(\theta - m(\theta))], \qquad \theta \in \{\underline{\theta}, \bar{\theta}\} \tag{3}$$

Definition 5 *A contract is* acceptable *to the financial intermediary if and only if*

$$\sum_{m\in\Theta}\Delta(m)\left[\sum_{\sigma\in\Theta\cup\emptyset}\Delta(\sigma|m,q(m))z(m,\sigma)-q(m)(\alpha+b)\kappa\right]\geq b\rho, \quad (4)$$

where $\Delta(\cdot)$ is an operator generating unconditional probability measures over its arguments. The state of nature θ is unobservable to the financial intermediary, therefore expectations in (4) are formed over the probability measure constructed over the entrepreneur's possible reports, which combines the likelihoods of shocks θ with the entrepreneur's ex post best response reporting strategy.

Definition 6 *A contract is* feasible *if and only if it is acceptable, and satisfies the budget constraints.*

Definition 7 *An incentive compatible contract is* efficient *if and only if it maximises the entrepreneur's utility subject to feasibility*

$$\max_{\Gamma}\sum_{\theta\in\Theta}\pi(\theta)\sum_{\sigma\in\Sigma\cup\emptyset}\Delta(\sigma|m,q(m))U(x(m,\sigma,\theta)) \quad (5)$$

subject to

$$(1),\ (2),\ (4),\quad q(m)\in[0,1].$$

Proposition 1 *In any efficient contract:*

1. *The financial intermediary's participation constraint (4) is binding,*
2. *high type reports are never audited, $q(\bar{m})=0$, and*
3. *the downward incentive compatibility constraint (equation 2, where $\theta=\bar{\theta}$) is binding when either (a) Type-I audit errors occur $(\eta(\underline{\theta})>0)$, or (b) utility is bounded below $(U(0)=0)$.*

A short description of the proposition follows, while Appendix A provides formal perturbation arguments for parts 2 and 3.

For part 1, note that if the intermediary's participation constraint were slack, repayments following high type reports $z(\bar{m}, \emptyset)$ could be reduced. That would increase the expected utility of the entrepreneur without breaching the incentive compatibility constraint.

The intuition for part 2 is as follows: Let it be the case that audits are required to prevent low type agents from declaring high type reports. Such a contract must be increasing rather than reducing consumption risk relative to some strictly superior non-contingent contract.

Proposition 1 part 3 shows that if high type entrepreneurs strictly prefer to report truthfully their income, then it must be the case that either high type consumption could be transferred to low states, or the auditing probability and expense could be reduced, allowing a direct increase in expected utility, or relaxing the participation constraint respectively. Note that the proof provided for Proposition 1 part 3(a) does not require the entrepreneur to freely choose the audit probability q. When audit signals are imperfect, the incentive compatibility constraint is binding even when all low reports are audited.

Corollary 1 *Let the audit probability q be constrained arbitrarily. Subject to this constraint, the downward incentive compatibility constraint (equation 2, where $\theta = \bar{\theta}$) is binding for any efficient contract when Type-I audit errors occur ($\eta(\underline{\theta}) > 0$).*

We can now re-write the problem as a Kuhn-Tucker problem:

$$\begin{aligned}
\mathcal{L} = {} & \bar{\pi}\, U(x(\bar{m}, \emptyset, \bar{\theta})) + \underline{\pi}(1 - q(\underline{m}))\, U(x(\underline{m}, \emptyset, \underline{\theta})) \\
& + \underline{\pi} q(\underline{m})(1 - \eta(\underline{\theta}))\, U(x(\underline{m}, \underline{\sigma}, \underline{\theta})) + \underline{\pi} q(\underline{m})\eta(\underline{\theta})\, U(x(\underline{m}, \bar{\sigma}, \underline{\theta})) \\
& + \lambda \left[\begin{array}{l} (\alpha + b)(\mathbb{E}(\theta) - \rho - \underline{\pi} q(\underline{m})\kappa) + \alpha\rho - \bar{\pi} x(\bar{m}, \emptyset, \bar{\theta}) - \underline{\pi}(1 - q(\underline{m})) x(\underline{m}, \emptyset, \underline{\theta}) \\ -\underline{\pi} q(\underline{m})(1 - \eta(\underline{\theta})) x(\underline{m}, \underline{\sigma}, \underline{\theta}) - \underline{\pi} q(\underline{m})\eta(\underline{\theta}) x(\underline{m}, \bar{\sigma}, \underline{\theta}) \end{array} \right] \\
& + \mu \left[\begin{array}{l} U(x(\bar{m}, \emptyset, \bar{\theta})) - (1 - q(\underline{m})) U[x(\underline{m}, \emptyset, \underline{\theta}) + (\alpha + b)(\bar{\theta} - \underline{\theta})] \\ -q(\underline{m})(1 - \eta(\bar{\theta})) U[x(\underline{m}, \bar{\sigma}, \underline{\theta}) + (\alpha + b)(\bar{\theta} - \underline{\theta})] \\ -q(\underline{m})\eta(\bar{\theta}) U[x(\underline{m}, \underline{\sigma}, \underline{\theta}) + (\alpha + b)(\bar{\theta} - \underline{\theta})] \end{array} \right] \\
& + \nu_0 q(\underline{m}) + \nu_1 (1 - q(\underline{m})). \qquad (6)
\end{aligned}$$

The Lagrange multipliers λ and μ are attached respectively to the participation constraint and the incentive compatibility constraint, and the Kuhn-

Tucker multipliers, ν_0 and ν_1, to the upper and lower bounds on the probability of audit respectively. Proposition 1 ensures that the participation and incentive compatibility constraints are binding under any efficient contract. The upper and lower bounds on the audit probability $q(\underline{m})$ are occasionally binding constraints.

The first order necessary conditions are described in detail as they will be used at various points to establish certain facts about efficient contracts. Hence:

$$x(\bar{m}, \emptyset, \bar{\theta}): \quad 0 = \bar{\pi}\, U'(x(\bar{m}, \emptyset, \bar{\theta})) - \lambda \bar{\pi} + \mu U'(x(\bar{m}, \emptyset, \bar{\theta})) \tag{6a}$$

$$\begin{aligned} x(\underline{m}, \emptyset, \underline{\theta}): \quad 0 &= \underline{\pi}(1 - q(\underline{m}))\, U'(x(\underline{m}, \emptyset, \underline{\theta})) - \lambda \underline{\pi}(1 - q(\underline{m})) \\ &\quad - \mu(1 - q(\underline{m}))U'[x(\underline{m}, \emptyset, \underline{\theta}) + (\alpha + b)(\bar{\theta} - \underline{\theta})] \end{aligned} \tag{6b}$$

$$\begin{aligned} x(\underline{m}, \underline{\sigma}, \underline{\theta}): \quad 0 &= \underline{\pi} q(\underline{m})(1 - \eta(\underline{\theta}))\, U'(x(\underline{m}, \underline{\sigma}, \underline{\theta})) - \lambda \underline{\pi} q(\underline{m})(1 - \eta(\underline{\theta})) \\ &\quad - \mu q(\underline{m})\eta(\bar{\theta})U'[x(\underline{m}, \underline{\sigma}, \underline{\theta}) + (\alpha + b)(\bar{\theta} - \underline{\theta})] \end{aligned} \tag{6c}$$

$$\begin{aligned} x(\underline{m}, \bar{\sigma}, \underline{\theta}): \quad 0 &= \underline{\pi} q(\underline{m})\eta(\underline{\theta})\, U'(x(\underline{m}, \bar{\sigma}, \underline{\theta})) - \lambda \underline{\pi} q(\underline{m})\eta(\underline{\theta}) \\ &\quad - \mu q(\underline{m})(1 - \eta(\bar{\theta}))U'[x(\underline{m}, \bar{\sigma}, \underline{\theta}) + (\alpha + b)(\bar{\theta} - \underline{\theta})] \end{aligned} \tag{6d}$$

$$\begin{aligned} b: \quad 0 &= \lambda(\mathbb{E}(\theta) - \rho - \underline{\pi} q \kappa) \\ &\quad - \mu(\bar{\theta} - \underline{\theta}) \left[\begin{array}{l} (1 - q(\underline{m}))U'[x(\underline{m}, \emptyset, \underline{\theta}) + (\alpha + b)(\bar{\theta} - \underline{\theta})] \\ + q(\underline{m})(1 - \eta(\bar{\theta}))U'[x(\underline{m}, \bar{\sigma}, \underline{\theta}) + (\alpha + b)(\bar{\theta} - \underline{\theta})] \\ + q(\underline{m})\eta(\bar{\theta})U'[x(\underline{m}, \underline{\sigma}, \underline{\theta}) + (\alpha + b)(\bar{\theta} - \underline{\theta})] \end{array} \right] \end{aligned} \tag{6e}$$

$$\begin{aligned} q(\underline{m}): \quad 0 &= -\underline{\pi}\, U(x(\underline{m}, \emptyset, \underline{\theta})) + \underline{\pi}(1 - \eta(\underline{\theta}))\, U(x(\underline{m}, \underline{\sigma}, \underline{\theta})) + \underline{\pi}\eta(\underline{\theta})\, U(x(\underline{m}, \bar{\sigma}, \underline{\theta})) \\ &\quad + \lambda \left[\underline{\pi} x(\underline{m}, \emptyset, \underline{\theta}) - \underline{\pi}(1 - \eta(\underline{\theta}))x(\underline{m}, \underline{\sigma}, \underline{\theta}) - \underline{\pi}\eta(\underline{\theta})x(\underline{m}, \bar{\sigma}, \underline{\theta}) \right] \\ &\quad + \mu \left[\begin{array}{l} + U[x(\underline{m}, \emptyset, \underline{\theta}) + (\alpha + b)(\bar{\theta} - \underline{\theta})] \\ - (1 - \eta(\bar{\theta}))U[x(\underline{m}, \bar{\sigma}, \underline{\theta}) + (\alpha + b)(\bar{\theta} - \underline{\theta})] \\ - \eta(\bar{\theta})U[x(\underline{m}, \underline{\sigma}, \underline{\theta}) + (\alpha + b)(\bar{\theta} - \underline{\theta})] \end{array} \right] \\ &\quad - \lambda(\alpha + b)\underline{\pi}\kappa + \nu_0 - \nu_1. \end{aligned} \tag{6f}$$

The general problem is non-convex, owing to the uncertainty faced by misreporting high type agents. Numerical results in the following sections indeed confirm that multiple locally efficient contracts may result.

4 PERFECT AUDITS

In the introduction we stated that the interaction between leverage and costly, *imperfect* audits underpins the optimality of deterministic contracts. Before establishing that, and other, results it is insightful to analyse the case of *perfect* audits. We find, as did Mookherjee and Png (1989), that debt contracts are not optimal. Moreover, we go on to show that optimal leverage is unbounded, absent other restrictions.

Theorem 1 *(Mookherjee and Png (1989)) When audits yield correct signals with certainty ($\eta(\underline{\theta}) = \eta(\bar{\theta}) = 0$), standard debt contracts are inefficient, $q^*(\underline{m}) \neq 1$.*

The proof proceeds as follows. First, Lemma 1 shows that when audits are perfect, any allocation which is feasible under a standard debt contract can be achieved while the incentive compatibility constraint is slack. Then, Proposition 1 part 3 shows that the downward incentive compatibility constraint cannot be slack under any efficient contract. Therefore, all allocations which are feasible under a standard debt contract are inefficient.

Lemma 1 *Let audits yield correct signals with certainty ($\eta(\underline{\theta}) = \eta(\bar{\theta}) = 0$), consumption be positive in all states $x(\cdot) > 0$, and the probability of audit of low type reports be equal to 1, $q(\underline{m}) = 1$. Any feasible allocation can be implemented with the incentive compatibility constraint slack.*

Proof. To prove Lemma 1, first re-write the downward incentive compatibility constraint (equation 2, where $\theta = \bar{\theta}$), with $q = 1$ as follows: $U(x(\bar{m}, \emptyset, \bar{\theta}) \geq U[(\alpha + b)\bar{\theta} - z(\underline{m}, \bar{\sigma})]$. Under perfect audits, the observable pair $(\underline{m}, \bar{\sigma})$ correctly identifies misreporting high type entrepreneurs with certainty. Under a truth-telling equilibrium, the repayment $z(\underline{m}, \bar{\sigma})$ is not made by any agent. Any feasible allocation can be perturbed by increasing $z(\underline{m}, \bar{\sigma})$, which does not affect the participation constraint of the financial intermediary, does not affect the ex ante welfare of the entrepreneur, but does ensure that the incentive compatibility constraint is slack. ■

Proposition 2 *When audits are perfect ($\eta(\underline{\theta}) = \eta(\bar{\theta}) = 0$), sufficiently inexpensive ($\kappa < (\mathbb{E}(\theta) - \rho)/\underline{\pi}$) and projects enjoy constant returns to scale, efficient leverage and entrepreneurial consumption are infinite.*

Proof. Set the probability of audit equal to one, $q = 1$. Substituting the budget constraints (1) into the participation constraint (4) shows that when $\kappa < (\mathbb{E}(\theta) - \rho)/\underline{\pi}$, expected consumption will be rising in b. Lemma 1 states that the incentive compatibility constraint need not bind for any allocation satisfying the participation constraint, given any level of borrowing b. ■

Under perfect audits, auditing with a high probability allows us to equate the entrepreneur's ex post marginal utility across all states, regardless of leverage. When audits are sufficiently inexpensive, higher leverage permits higher entrepreneurial consumption in all states. Leverage in equilibrium is only bound by decreasing technological returns to scale, as in Gale and Hellwig (1985), or through general equilibrium effects.

If the probability of audit is sufficiently high, large penalties charged against misreporting high type entrepreneurs ensure that the incentive compatibility constraint is slack for any schedule of positive consumption allocations earned with positive probability. Any further audits would be wasteful, as the resource costs of additional audits would tighten the participation constraint of the fiinancial intermediary, and as the incentive compatibility constraint was already slack, no further risk sharing gains would be available from the additional audits.

Efficient allocations require that any agent who earns a low type return, declares their return truthfully, yet receives a high type audit signal $(\underline{m}, \bar{\sigma}, \underline{\theta})$ should face a repayment greater than their revenue $z(\underline{m}, \bar{\sigma}) > (\alpha + b)\underline{\theta}$. However, this outcome occurs with zero probability when audits are perfect.

5 Imperfect audits

When type-I errors occur with positive probability ($\eta(\underline{\theta}) > 0$), Lemma 1 and subsequently Theorem 1 cease to hold; the outcome $(\underline{m}, \bar{\sigma}, \underline{\theta})$ occurs with positive probability in any contract with auditing $(q^*(\underline{m}) > 0)$. Even if the audit probability is high, a further increase in the audit probability does increase the set of feasible consumption allocations available to the entrepreneur. Increasing the probability of audit under imperfect audits allows the incentive costs of contract enforcement to be defrayed more widely, increasing risk sharing across states. If audit costs are low, then increasing the probability of audit is worthwhile even when the probability of audit is already high.

Theorem 2 *Let borrowing be taken as given $b = \hat{b}$. When type-I audit errors occur with positive probability ($\eta(\underline{\theta}) > 0$), there exists some strictly positive audit cost $\hat{\kappa}$ such that for all $\kappa < \hat{\kappa}$, standard debt contracts ($q(\underline{m}) = 1$) are efficient.*

Proof. We consider an arbitrary efficient contract with interior audit probability $q(\underline{m}) \in (0, 1)$, and show from the first order condition for the probability of audit $\mathcal{L}_{q(\underline{m})}$ that if audit costs were sufficiently low, the initial contract could be strictly improved by an increase in audit probability $q(\underline{m})$.

To simplify notation, in this section we will define $B(b) := (\alpha + b)(\bar{\theta} - \underline{\theta})$. Also, as we are only considering allocations consistent with truth-telling, we will drop the report variable from the consumption allocation $x(\sigma, \underline{\theta}) := x(\underline{m}, \sigma, \underline{\theta})$.

Consider the first order necessary condition for $q(\underline{m})$ (6f), which can be re-written as follows:

$$\begin{aligned}
\mathcal{L}_{q(\underline{m})}: \quad 0 = {} & \underline{\pi}(1 - \eta(\underline{\theta}))\,[U(x(\underline{\sigma}, \underline{\theta})) - \lambda x(\underline{\sigma}, \underline{\theta})] - \mu\eta(\bar{\theta})U[x(\underline{\sigma}, \underline{\theta}) + B(b)] \\
& + \underline{\pi}\eta(\underline{\theta})\,[U(x(\bar{\sigma}, \underline{\theta})) - \lambda x(\bar{\sigma}, \underline{\theta})] - \mu(1 - \eta(\bar{\theta}))U[x(\bar{\sigma}, \underline{\theta}) + B(b)] \\
& - \underline{\pi}\,[U(x(\emptyset, \underline{\theta})) + \lambda x(\emptyset, \underline{\theta})] + \mu\, U[x(\emptyset, \underline{\theta}) + B(b)] \\
& - \lambda(\alpha + b)\underline{\pi}\kappa + \nu_0 - \nu_1.
\end{aligned} \tag{7}$$

Up to division by $q(\underline{m})$, the consumption variables $x(\underline{\sigma}, \underline{\theta}), x(\bar{\sigma}, \underline{\theta})$ enter $\mathcal{L}_{q(\underline{m})}$ in the same way that they enter the entrepreneur's problem $\mathcal{L}$ (equation 6). This means that the first order necessary conditions for $x(\underline{\sigma}, \underline{\theta}), x(\bar{\sigma}, \underline{\theta})$ (equations 6c and 6d respectively), also identify a stationary point of $\mathcal{L}_{q(\underline{m})}$, with respect to the consumption allocations of audited agents ($x(\underline{\sigma}, \underline{\theta}), x(\bar{\sigma}, \underline{\theta})$) and holding other variables constant.

This property has a straightforward economic interpretation: efficiently allocating consumption to audited agents, is the same problem as maximising the gain from additional audits, which is expressed by the first order condition $\mathcal{L}_{q(\underline{m})}$.

We can think of the first three lines of (7) as the gains attained from the information provided by a marginal increase in the probability of audit. The fourth line contains the marginal resource cost, plus Kuhn-Tucker multipliers associated with the upper and lower bounds on the audit probability.

Here it is important that audits are imperfect, which by Corollary 1 ensures that the incentive compatibility constraint is binding, and $\mu > 0$, regardless of $q(\underline{m})$. Were audits perfect, sufficiently high audit probabilities would result in slackness in the incentive compatibility constraint ($\mu = 0$), leaving the first order conditions for $x(\underline{\sigma}, \underline{\theta}), x(\emptyset, \underline{\theta})$ equated.

Consider the allocation $\hat{x}(\underline{\sigma}, \underline{\theta}) = \hat{x}(\bar{\sigma}, \underline{\theta}) = x(\emptyset, \underline{\theta})$. This allocation would leave the sum of the first three lines of (7) equal to zero. But this allocation is not a stationary point of $\mathcal{L}_{q(\underline{m})}$, and does not satisfy the first order necessary conditions for $x(\underline{\sigma}, \underline{\theta}), x(\bar{\sigma}, \underline{\theta})$ (equations 6c and 6d respectively).

We can do better by decreasing $x(\bar{\sigma}, \underline{\theta})$, which has a low weight in expected welfare and a high weight in the incentive compatibility constraint, and increasing $x(\underline{\sigma}, \underline{\theta})$, which has a relatively high weight in expected welfare and a low weight in the incentive compatibility constraint. This perturbation would leave the sum of the first three lines of (7) strictly greater than zero, such that for sufficiently low audit costs κ, additional audits would always be welfare enhancing. ■

Theorem 2 shows that standard debt can be efficient under imperfect audits. The remainder of this section explores the global efficiency of standard debt, and the quantitative relevance. As we will see, an important determinant of the efficiency of standard debt will be whether the entrepreneur has access to a leverage margin—enabling them to scale up and down the size of the project ex ante.

Under perfect audits, proposition 2 showed that when audit costs are low, constant technological returns to scale would result in unbounded leverage and entrepreneurial consumption. When type-I audit errors occur with positive probability, that result no longer holds. To see this, note that the incentive compatibility constraint (3) ensures that for some $\sigma \in \{\emptyset, \underline{\sigma}, \bar{\sigma}\}$, $x(\bar{m}, \emptyset, \bar{\theta}) - x(\underline{m}, \sigma, \underline{\theta}) \geq (\alpha + b)(\bar{\theta} - \underline{\theta})$. Combining this with assumption 2 ensures that as b increases, consumption risk must be increasing and consumption in some state $x(\underline{m}, \sigma, \underline{\theta})$ must tend toward zero.

Entrepreneurs will not choose contracts with consumption bundles too close to zero, where their marginal utility of consumption tends to infinity. Entrepreneurs' aversion to low consumption bundles in bad states encourages them to choose contracts with limited leverage, even when their project enjoys constant technological returns to scale. This argument is formalised in the following proposition.

Proposition 3 *When type-I errors occur with positive probability ($\eta(\underline{\theta}) > 0$), positive entrepreneurial consumption in all states requires that high type consumption, leverage and the probability of audit satisfy the following inequalities*

a. *high type consumption,* $x(\bar{m}, \emptyset, \bar{\theta}) > (\alpha + b)(\bar{\theta} - \underline{\theta})$,

b. *leverage,* $\dfrac{\alpha + b}{\alpha} < \dfrac{\rho}{\rho - \underline{\theta} + \underline{\pi} q \kappa}$, *and*

c. *the probability of audit,* $q < \min\left(1, \dfrac{1}{\underline{\pi}\kappa} \dfrac{\alpha\rho}{\alpha + b}\left[1 - \dfrac{\alpha + b}{\alpha} \cdot \dfrac{\rho - \underline{\theta}}{\rho}\right]\right)$.

Proof. The first part of the proposition is a direct weakening of equation 3, which is presented in a form such that all consumption bundles contained in the constraint are earned with positive unconditional probability in contracts with auditing, and are therefore positive by the assumption specified in the proposition.

$$x(\bar{m}, \emptyset, \bar{\theta}) > (\alpha + b)(\bar{\theta} - \underline{\theta}) \tag{8}$$

Substituting equation 8 and the budget constraints (1) into the participation constraint (4) with the assumption that entrepreneurial consumption is positive in every state yields the following inequality:

$$(\alpha + b)(\bar{\pi}\bar{\theta} + \underline{\pi}\underline{\theta} - \rho - \underline{\pi} q \kappa) + \alpha\rho > \bar{\pi}(\alpha + b)(\bar{\theta} - \underline{\theta}) \tag{9}$$

which can be rearranged to confirm parts (b) and (c) of the proposition. ■

We proceed allowing borrowing b to be chosen freely, under the assumption of constant technological returns to scale. This does not mean that firms enjoy constant returns to scale. Firm size is endogenously bounded above according to proposition 3. It does mean that the only source of decreasing returns to scale is the information asymmetry between the entrepreneur and external finance providers.

Figure 1 presents evidence of the quantitative relevance of standard debt in our framework for a sample parameterisation. Along the horizontal axis, the risk of the entrepreneur's project $(\bar{\theta} - \underline{\theta})$ is increasing, holding expected returns $(\mathbb{E}(\theta))$ constant. The vertical axis plots audit costs as a share of total assets under management. Two features of the simulation are striking: First, standard debt $(q^* = 1)$ is very prevalent. Very low project risk or high audit costs are required for standard debt to be inefficient. Second, stochastic audit regimes $(0 < q^* < 1)$ are rare. Indeed, when risk is low, efficient contracts 'jump' from standard debt $(q^* = 1)$ to non-contingent debt contracts $(q^* = 0)$. In the model, there is no cost associated with writing a 'complex' contract with stochastic audit regimes, as are optimal under the perfect audits framework. Yet, entrepreneurs tend to prefer 'simple', non-contingent or standard debt contracts.

When contracts are constrained by the upper and lower bound on audits ($q^* = 0$ *or* 1), local analysis of the entrepreneur's problem is relatively straightforward, yielding closed form solutions under constant relative risk aversion when the likelihood of type-II audit error is zero ($\eta(\bar{\theta}) = 0$):[9]

Proposition 4 *When the likelihood of type-I and type-II errors are positive and zero respectively (*$\eta(\underline{\theta}) > 0, \eta(\bar{\theta}) = 0$*) and preferences exhibit constant relative risk aversion* $U(x) = x^{1-\gamma}/(1-\gamma)$*, leverage, consumption allocations and shadow prices of standard debt contracts and non-contingent debt contracts can be represented by closed-form expressions in terms of exogenous parameters.*

[9]Type-II errors, while perhaps more familiar than type-I errors, have little effect on the nature of efficient contracts if they occur with a low probability. The implications of type-II errors are investigated in appendix D.

The proof of proposition 4 is given in appendix B. Displayed below are solutions to standard debt contracts under logarithmic utility ($U(x) = \log x$).

$$x(\bar{m}, \emptyset, \bar{\theta}) = \alpha\rho \, \frac{1}{1-\zeta}, \qquad x(\underline{m}, \underline{\sigma}, \underline{\theta}) = \alpha\rho, \qquad x(\underline{m}, \bar{\sigma}, \underline{\theta}) = \alpha\rho \, \frac{\underline{\pi}\eta(\underline{\theta})}{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})},$$

$$b = \frac{\alpha\rho}{\bar{\theta} - \underline{\theta}} \, \frac{\zeta}{1-\zeta} \left(\frac{\bar{\pi} + \underline{\pi}\eta(\underline{\theta})}{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})} \right) - \alpha, \qquad \text{where} \qquad \zeta = \frac{\mathbb{E}(\theta) - \rho - \underline{\pi}\kappa}{\bar{\pi}(\bar{\theta} - \underline{\theta})}. \tag{10}$$

From the solutions presented in (10), we can derive measures of leverage and loan coupon rates, which are more easily observed than entrepreneurs' consumption allocations in practise. Leverage, l, as measured by the total assets managed by the entrepreneur over their initial net worth can be described as follows:

$$l = \frac{\alpha + b}{\alpha} = \frac{\rho}{\bar{\theta} - \underline{\theta}} \, \frac{\zeta}{1-\zeta} \left(\frac{\bar{\pi} + \underline{\pi}\eta(\underline{\theta})}{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})} \right). \tag{11}$$

We can determine the net interest (coupon) rate on loans, r, by subtracting one from the ratio of the repayment following high reports $z(\bar{m}, \emptyset)$ and the initial amount borrowed b. Combining the budget constraints with (10) yields

$$r = \frac{z(\bar{m}, \emptyset)}{b} - 1 = (\rho - 1) + \rho\zeta \, \frac{(\bar{\theta} - \rho)(\bar{\pi} + \underline{\pi}\eta(\underline{\theta})) - (\bar{\theta} - \underline{\theta})(\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta}))}{\rho\zeta(\bar{\pi} + \underline{\pi}\eta(\underline{\theta})) - (1-\zeta)(\bar{\theta} - \underline{\theta})(\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta}))}. \tag{12}$$

The first term on the right hand side, $(\rho - 1)$, is the opportunity cost of funds for the financial intermediary, expressed as a net interest rate. The second term captures the interest rate credit spread, as measured by the difference between the loan interest (coupon) rate and the financial intermediary's opportunity cost of funds. Section 6 presents example comparative statics for leverage ratios and the loan interest rate.

The solutions obtained by proposition 4 are local, though the entrepreneur's problem exhibits 'jumps' between locally efficient contracts. Fact 1 states that for a standard debt contract to be globally efficient, it must be both locally efficient, and superior to any non-contingent contract. While these two necessary conditions do not rule out an alternative globally efficient contract, we have not been able to find a numerical example where the two necessary conditions expressed in fact 1 are satisfied, and standard debt contracts are not globally efficient.

Fact 1 *Let* Γ *be a globally efficient contract, and* $q(\Gamma) = 1$. *(a) The Kuhn-Tucker multiplier on the constraint* $q \leq 1$ *must be positive* $(\nu_1 > 0)$*, and (b) expected utility under* Γ *must exceed the maximum utility attainable conditional upon private information (*$\mathbb{E}(U|\Gamma) \geq \mathbb{E}(U|(\Gamma^*|q = 0))$*).*

When utility is CRRA and type-II errors do not occur $(\eta(\bar{\theta}) = 0)$, then by proposition 4 we can check fact 1 part (b) directly from the closed-form solutions to standard and non-contingent debt contracts provided in appendix B. When utility is logarithmic, we can also directly check fact 1 part (a) by the following result:

Proposition 5 *When utility is logarithmic, and type-II errors do not occur* $(\eta(\bar{\theta}) = 0)$*, the Kuhn-Tucker multipliers on the upper and lower bounds for the audit probability,* $0 \leq q \leq 1$*, can be described by closed-form expressions.*

A derivation of proposition 5 is provided in Appendix B.3.

We now return to reconsider the bang-bang feature of efficient contracts observed in figure 1, which we formalise by proposition 6.

Proposition 6 *When audits are imperfect, there exist parameter specifications which permit both non-contingent* $(q = 0)$ *and standard* $(q = 1)$ *debt contracts as locally efficient contracts.*

When project risk and audit costs are low, efficient contracts appear to jump from non-contingent debt to standard debt. Figure 2 plots an example of this bang-bang behaviour. In figure 2, the determination of efficient contracts is deconstructed by leverage and audit strategy for one example parameter specification. The horizontal axis plots levels of borrowing. The lower panel plots the efficient probability of audit, conditional upon borrowing, and the upper panel plots the attainable expected welfare conditional upon borrowing. The solid line plots expected welfare attainable with a non-contingent contract ($q = 0$), the dashed line allows the audit strategy to be chosen optimally.

The efficiency of standard debt is sensitive to the assumption that the entrepreneur can determine the scale of the project. When leverage is low, total risk is low, and the gains from insurance provided by auditing are low. On the other hand, when leverage is high, proposition 3 showed that auditing will push the minimum consumption allocation closer to zero. Audits are only useful to the extent that the entrepreneur can absorb type-I errors in low states.

Appendix C solves an example where non-contingent and standard debt contracts are locally efficient. Under the non-contingent debt contract, the marginal resource cost of additional audits exceeds the gains obtained from the audit signal. Under the standard debt contract, the marginal resource gain from reducing the audit probability is smaller than the cost of foregoing the information and incentive gains obtained via the marginal audit.

Leverage is higher under the non-contingent contract than under the efficient standard debt contract, and therefore the marginal resource cost of audits is higher than under the standard debt contract. The marginal benefit from information obtained in additional audits is actually identical under the two locally efficient contracts considered. Under the non-contingent contract, the difference in expected marginal utility across project outcomes is high, suggesting that the gains from insurance should be higher than under the efficient standard debt contract. However, low consumption of low type entrepreneurs also makes type-I errors particularly costly, preventing significant penalties in auditing contracts, and reducing the benefits obtained by auditing.

6 Comparative statics

The preceding paragraphs explained how small parameter changes—for example a small increase in project risk—can cause the efficient contract to jump from a high leverage, low risk premium, non-contingent contract, to a low leverage, high risk premium and highly contingent standard debt contract.

Within parameter neighbourhoods where defautable debt contracts are efficient, we can use the solutions obtained in appendix B to analyse local perturbations to expected returns, risk, audit costs and audit quality. Consider the following parameterisation: The probability of default is $\underline{\pi} = 1/10$; conditional upon realisation of the low state, the audit signal returns a high state with vanishing probability $\eta(\underline{\theta}) \to 0^+$; the gross opportunity cost of funds $\rho = 21/20$, equivalent to a 5% interest rate; the expected gross return on projects $\mathbb{E}(\theta) = 6/5$; the coefficient of relative risk aversion $\gamma = 1$; audit costs as a share of the initial assets devoted to the project are $\kappa = 9/80$, and in low states, the project returns $\underline{\theta} = 33/40$. Subsequently, the high type return is $\bar{\theta} = \frac{1}{\bar{\pi}}[\mathbb{E}(\theta) - \underline{\pi\theta}] = 149/120$; project risk is $(\bar{\theta} - \underline{\theta}) = 5/12$; and $\zeta = [\mathbb{E}(\theta) - \rho - \underline{\pi}\kappa]/[\bar{\pi}(\bar{\theta} - \underline{\theta})] = 37/100$.

By equation 11, leverage (l) is equal to

$$l = \frac{\rho}{\bar{\theta} - \underline{\theta}} \frac{\zeta}{1 - \zeta} \left(\frac{\bar{\pi} + \underline{\pi}\eta(\underline{\theta})}{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})} \right) = 4.$$

By equation 12, the loan interest (coupon) rate is

$$r = (\rho - 1) + \rho\zeta \, \frac{(\mathbb{E}(\theta) + \underline{\pi}(\bar{\theta} - \underline{\theta}) - \rho)(\bar{\pi} + \underline{\pi}\eta(\underline{\theta})) - (\bar{\theta} - \underline{\theta})(\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta}))}{\rho\zeta(\bar{\pi} + \underline{\pi}\eta(\underline{\theta})) - (1 - \zeta)(\bar{\theta} - \underline{\theta})(\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta}))} = 10\%,$$

We can take derivatives of the log of the leverage ratio to find the semi-elasticities of leverage with respect to expected returns, the intermediary's opportunity cost, risk, audit costs and type-I errors:

$$\frac{d\log l}{d(\mathbb{E}(\theta))} = 4.23, \qquad \frac{d\log l}{d\rho} = -3.28, \qquad \frac{d\log l}{d(\bar{\theta} - \underline{\theta})} = -3.81,$$

$$\frac{d\log l}{d\kappa} = -0.423 \qquad \text{and} \qquad \frac{d\log l}{d\eta(\underline{\theta})} = -0.189$$

respectively, reported to 3 significant figures.[10]

We can also determine the sensitivity of efficient loan interest rates to

[10]For example, an increase in expected returns by 0.01 causes an increase in the optimal leverage ratio by 4.23 percent. Note that rather than reporting semi-elasticities for returns in each state $(\bar{\theta}, \underline{\theta})$, we have reported responses to expected returns $\mathbb{E}(\theta)$ and risk $(\bar{\theta} - \underline{\theta})$, which we find to be more useful for intuition.

underlying parameters:

$$\frac{dr}{d(\mathbb{E}(\theta))} = -0.219, \qquad \frac{dr}{d\rho} = 1.20, \qquad \frac{dr}{d(\bar{\theta} - \underline{\theta})} = 0.197,$$

$$\frac{dr}{d\kappa} = 0.155 \qquad \text{and} \qquad \frac{dr}{d\eta(\underline{\theta})} = -0.0357.$$

Whilst these calculations are specific to our example, they do provide insights into the tradeoffs more generally faced by entrepreneurs.

An increase in expected project returns $\mathbb{E}(\theta)$ increases leverage and, perhaps surprisingly, decreases loan interest rates. The prospect of higher returns encourages entrepreneurs to increase leverage, but they are limited in doing so due to the presence of type-I errors. A decrease in interest payments in high states r leaves the entrepreneur enough funds in low states to repay loans in full following type-I errors. In order to satisfy the financial intermediary's participation constraint, providing an expected return of ρ on loans b net of auditing costs, the entrepreneur must absorb more project risk. Low state repayments following verified reports $(z(\underline{m}, \underline{\sigma})/b)$ are increased. Decreases in project risk $(\bar{\theta} - \underline{\theta})$ and audit costs (κ) have a similar effect on leverage and interest rates as increases in expected returns $\mathbb{E}(\theta)$.

Following a decrease in the financial intermediary's opportunity cost of funds, loan interest rates fall by an even greater amount. In other words, the spread between loan interest rates and the opportunity cost of funds is increasing in the opportunity cost of funds. First, a decrease in ρ allows the entrepreneur to make lower repayments in all states, while meeting the intermediary's participation constraint. Second, when ρ is low, the entrepreneur enjoys more of the gains from increased leverage. As in the case of an increase in expected returns $\mathbb{E}(\theta)$, increases in leverage require the entrepreneur to further lower interest repayments such that full repayments are possible even following type-I errors. To compensate, repayments following verified low reports $(z(\underline{m}, \underline{\sigma})/b)$ must increase.

The only variable which moves leverage and loan interest rates in the same direction is audit quality, as measured by the conditional probability of type-I error $(\eta(\underline{\theta}))$. An increase in the probability of error encourages entrepreneurs to increase consumption in the unlikely event of a type-I error.

This adjustment is achieved first by decreasing leverage, which at any given interest rate increases the amount of resources remaining following type-I errors, and secondly by decreasing the loan interest rate r, which further reduces repayments following type-I errors. This adjustment requires an increase in repayments in low states ($z(\underline{m}, \underline{\sigma})/b$) to compensate the financial intermediary.

7 A FOUR-STATE EXAMPLE

In this section we extend the model by increasing the number of states from two to four. Unfortunately, that makes the model analytically intractable.[11] The purpose of this extension is twofold. First, it is clearly of interest to investigate how general our analytical results are likely to be, concerning the desirability of standard debt. Second, and related, it is of interest to compare the imperfect audits case to that of the perfect audits case of Mookherjee and Png (1989).

Typically, standard debt is defined as a contract where reports below some cutoff are audited with certainty. See, for example Townsend (1979). Audits above the cutoff are not audited. Thus, we state:

Definition 8 *In a model with $n > 2$ possible states ($\theta \in (\theta_1, \theta_2, ..., \theta_n)$ and $\theta_i < \theta_{i+1}$), a contract is a standard debt contract if and only if*

$$\exists\, K \in \{1, 2, ..., n\} \quad s.t. \quad q(m_j) = 1 \quad \forall j < K, \quad q(m_j) = 0 \quad \textit{otherwise.} \tag{13}$$

Note that incentive compatibility requires that all repayments following reports above the cutoff must be identical

$$z(m_i, \emptyset) = z(m_j, \emptyset) \quad \forall i, j \geq K. \tag{14}$$

Definition 8 generalises definition 3(b) used to analyse the two-state model in earlier sections. An important feature of standard debt is that when income is sufficiently high, repayments are not sensitive to income—the bor-

[11]Even with just four states, the problem contains 24 choice variables. We have been unable to find superior contracts to the examples presented below, but we do not prove that the contracts presented are globally efficient.

rower need not ever pay more than the coupon plus the principal. This is formalised in equation 14, and is not observable in the two-state model, where there is only one ‘high’ state. In this section with a multiple states model it turns out that imperfect audits do indeed motivate that feature of standard debt.

Figure 3 presents a numerically-solved example of locally efficient contracts under perfect and imperfect audits. In order to compare the incentive regimes under the two environments, borrowing b is set exogenously. The upper panel presents the probability distribution from which states are drawn. The second panel presents the expected repayment conditional upon the true state being equal to θ. The contract with signal errors (marked by $\times$) features constant repayments across the three higher states, and a reduced expected repayment in the lowest state. The perfect audits contract (marked by $+$) exhibits sharply increasing repayments across states—similar to an equity contract with variable dividends. The third panel presents the expected utility of the borrower, conditional upon the realised state. The contract with signal errors exhibits significant sensitivity between expected utility and project outcomes, across all states. The contract with perfect audits exhibits increasing expected utility across states, although the sensitivity of utility across states is very low. The fourth panel presents the auditing regime under each contract. The contract with signal errors resembles a standard debt contract: reports of the lowest state are followed by certain audits ($q(m_1) = 1$). Reports of any higher states are not audited ($q(m_2) = q(m_3) = q(m_4) = 0$). Under perfect audits, audits are conducted with low probability across all of the three lower states.

8 DISCUSSION

Standard debt contracts can be the optimal form of external finance contracts when contract enforcement is uncertain due to noisy audit signals. Supporting truth-telling under a stochastic audit strategies requires large penalties. When there is no guarantee that these penalties are fairly applied, these contracts will not be acceptable to risk averse entrepreneurs. The resulting efficient contracts will audit consequent only on low reports, but will likely audit low reports with certainty. As a result, only small penalties are required to ensure truth-telling in equilibrium. In fact, the penalty following a disputed report in an optimal debt contract is typically very close to fully repaying the debt.

Imperfect verification also implies other interesting properties of optimal contracts. For instance, it means that borrowers can only pass a limited amount of risk on to lenders, regardless of contracted audit strategies. And even when projects enjoy constant returns to scale and audits are relatively inexpensive, firm size and leverage is endogenously limited by the entrepreneur's risk preference.

We end with a final observation. The standard debt contracts derived under imperfect monitoring enjoy an additional benefit—one that we did not formalise. When enforcement is certain, or near certain, incentive compatibility is not sensitive to the risk tolerance of the entrepreneur. That reduces the potential for adverse selection in two forms: First, the preferences of the entrepreneur may be unobservable; and second, the entrepreneur may have access to hidden wealth. The presence of either of these sources of asymmetric information would make it more difficult to employ a stochastic incentive scheme.

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A PROOF OF PROPOSITION 1

Proof.

2. It is established that if it were the case that low type agents were indifferent to reporting high or low type messages, then the contract in place must be weakly inferior than a simple non-contingent debt contract. This argument is made via three perturbations which leave the expected utility of the entrepreneur either unchanged or increased, and relax the intermediary's participation constraint.

   Let the probability of audit following high type reports be positive, and the incentive compatibility constraint be binding for low type entrepreneurs: $q(\bar{m}) > 0$ and

$$\sum_{\sigma \in \Theta \cup \emptyset} \Delta(\sigma(\underline{\theta}, q(\underline{m})))U[(\alpha + b)\underline{\theta} - z(\underline{m}, \sigma(\underline{\theta}, q(\underline{m})))] = \sum_{\sigma \in \Theta \cup \emptyset} \Delta(\sigma(\underline{\theta}, q(\bar{m})))U[(\alpha + b)\underline{\theta} - z(\bar{m}, \sigma(\underline{\theta}, q(\bar{m})))]. \tag{15}$$

   Perturbation 1: First replace all $z(\bar{m};\sigma)$ with $z'(\bar{m})$, such that

$$\sum_{\sigma \in \Theta \cup \emptyset} \Delta(\sigma(\bar{\theta}, q(\bar{m})))U[(\alpha + b)\bar{\theta} - z(\bar{m}, \sigma(\bar{\theta}, q(\bar{m})))] = U[(\alpha + b)\bar{\theta} - z'(\bar{m})].$$

The perturbation leaves truth-telling high type entrepreneurs indifferent. By Jensen's inequality, that perturbation will relax the intermediary's participation constraint, but could possibly violate the incentive compatibility constraint (15). If (15) is now not violated, then the proof is complete, and $q(\bar{m})$ can be set to zero, as the information yielded by auditing high type messages is ignored.

If (15) is violated after the perturbation, then

$$U[(\alpha+b)\underline{\theta} - z'(\bar{m})] > \sum_{\sigma\in\Theta\cup\emptyset} \Delta(\sigma(\underline{\theta}, q(\underline{m})))U[(\alpha+b)\underline{\theta} - z(\underline{m}, \sigma(\underline{\theta}, q(\underline{m})))].$$

Perturbation 2: Now replace $z(\underline{m}, \sigma)$ with $z'(\underline{m})$, such that

$$U[(\alpha+b)\underline{\theta} - z'(\underline{m})] = \sum_{\sigma\in\Theta\cup\emptyset} \Delta(\sigma(\underline{\theta}, q(\underline{m})))U[(\alpha+b)\underline{\theta} - z(\underline{m}, \sigma(\underline{\theta}, q(\underline{m}))).$$

By Jensen's inequality, that perturbation would also relax the participation constraint. Given that (15) is violated, $z'(\underline{m}) > z'(\bar{m})$: Low type entrepreneurs have an incentive to report high type shocks.

Perturbation 3: Replace $z'(\underline{m}), z'(\bar{m})$ with $z'' = \underline{\pi}z'(\underline{m}) + \bar{\pi}z'(\bar{m})$

Perturbation 3 restores incentive compatibility, by equating repayments across reports and states. The participation constraint is respected, as expected repayments are unchanged. The new contract offers expected utility which is strictly greater than under the original contract, by Jensen's inequality:

$$\bar{\pi}U[(\alpha+b)\bar{\theta}-z'']+\underline{\pi}U[(\alpha+b)\underline{\theta}-z''] > \bar{\pi}U[(\alpha+b)\bar{\theta}-z'(\bar{m})]+\underline{\pi}U[(\alpha+b)\underline{\theta}-z'(\underline{m})].$$

Audits are not required, as repayments are non-contingent.

3(a). For the incentive compatibility constraint (2 ; $\theta = \bar{\theta}$) to be satisfied, there must be some consumption bundle $x(\underline{m}, \sigma, \underline{\theta}) < x(\bar{m}, \emptyset, \bar{\theta})$, where $\sigma \in \{\emptyset, \bar{\sigma}, \underline{\sigma}\}$, and $x(\underline{m}, \sigma, \underline{\theta})$ is a bundle consumed with non-zero unconditional probability ($\Delta(\sigma|\underline{m}, q(\underline{m})) > 0$). Let equation (2 ; $\theta = \bar{\theta}$) be slack. There must be some $\varepsilon \in (0, \infty)$ such that a perturbation increasing $z(\bar{m}, \emptyset)$ by $\dfrac{\varepsilon}{\Delta(\emptyset|\bar{m}, 0)}$, and decreasing $z(\underline{m}, \sigma)$ by $\dfrac{\varepsilon}{\Delta(\sigma|\underline{m}, q(\underline{m}))}$, which would violate neither the participation nor the incentive compatibility constraints. This perturbation would increase expected utility by Jensen's inequality.

3(b). When audits are perfect, we cannot directly follow the proof of part 3(a)—the incentive compatibility constraint does not directly ensure that consumption x varies across states. First, if x does vary across states and audit signals, then we can follow the same argument as in part 3(a), and perturb toward a contract with less consumption variability. If x is constant across all states, then we could reduce the audit probability q, relaxing the participation constraint without violating the incentive compatibility constraint.

■

B EFFICIENT ALLOCATIONS UNDER NON-CONTINGENT AND STANDARD DEBT CONTRACTS

Here we solve for efficient allocations and borrowing when efficient audit strategies are deterministic, ie. when $q = 0$ *or* 1. Here, we assume CRRA utility, $U(x) = x^{1-\gamma}/(1-\gamma)$. In the text, we refer to the more tractable case of logarithmic utility, which can be found by setting $\gamma = 1$ in any of the solutions contained in this section.

The general problem outlined by equation 6 is non-convex, due to the presence of the lottery in the right hand side of the incentive compatibility constraint. When the probability of audit $q(\underline{m})$ is constrained by either its upper or lower bound, ν_0 or $\nu_1 > 0$, and the probability of type-II audit errors is zero $\eta(\bar{\theta}) = 0$, then locally the problem is convex, and we can use the first order approach to find local maxima.

B.1 PRIVATE INFORMATION CONTRACTS $(q = 0)$

When audits are not used ($q = 0$), efficient contracts are non-contingent. Repayments are independent of entrepreneurs' reports, and no audit signals are obtained to condition repayments. This certainty of repayment makes the incentive compatibility constraint linear in the choice variables, enabling us to solve the entrepreneur's problem with a Lagrangian:

$$\begin{aligned}\mathcal{L}_0 = {} & \bar{\pi}\, U(x(\bar{m},\emptyset,\bar{\theta})) + \underline{\pi}\, U(x(\underline{m},\emptyset,\underline{\theta})) \\ & + \lambda[(\alpha+b)(\mathbb{E}(\theta)-\rho) + \alpha\rho - \bar{\pi}x(\bar{m},\emptyset,\bar{\theta}) - \underline{\pi}x(\underline{m},\emptyset,\underline{\theta})] \\ & + \mu\left[U(x(\bar{m},\emptyset,\bar{\theta})) - U[x(\underline{m},\emptyset,\underline{\theta}) + (\alpha+b)(\bar{\theta}-\underline{\theta})]\right]. \end{aligned} \tag{16}$$

The first order conditions are

$$\begin{array}{lcccl} x(\bar{m},\emptyset,\bar{\theta}) & : & 0 & = & \bar{\pi}U'(x(\bar{m},\emptyset,\bar{\theta})) - \bar{\pi}\lambda + \mu\, U'(x(\bar{m},\emptyset,\bar{\theta})) \\ x(\underline{m},\emptyset,\underline{\theta}) & : & 0 & = & \underline{\pi}U'(x(\underline{m},\emptyset,\underline{\theta})) - \underline{\pi}\lambda - \mu\, U'[x(\underline{m},\emptyset,\underline{\theta}) + (\alpha+b)(\bar{\theta}-\underline{\theta})] \\ b & : & 0 & = & \lambda(\mathbb{E}(\theta)-\rho) - \mu(\bar{\theta}-\underline{\theta})U'[x(\underline{m},\emptyset,\underline{\theta}) + (\alpha+b)(\bar{\theta}-\underline{\theta})]. \end{array}$$

Substituting the incentive compatibility constraint and utility function into the first order conditions yields

$$\frac{U'(x(\underline{m},\emptyset,\underline{\theta}))}{U'(x(\bar{m},\emptyset,\bar{\theta}))} = \frac{\bar{\pi}}{\underline{\pi}}\ \left(\frac{\bar{\theta}-\rho}{\rho-\underline{\theta}}\right). \tag{17}$$

The right hand side of equation 17 shows that the ratio of weighted returns in high and low states can be interpreted as the cost of consumption in low states relative to consumption in high states. Equation 17 along with the incentive compatibility constraint can be substituted into the participation constraint to solve first for $x(\bar{m},\emptyset,\bar{\theta})$ and the remaining choice variables:

$$\begin{aligned} x(\bar{m},\emptyset,\bar{\theta}) &= \frac{\alpha\rho\,(\bar{\theta}-\underline{\theta})}{\left(\frac{\underline{\pi}(\rho-\underline{\theta})}{\bar{\pi}(\bar{\theta}-\rho)}\right)^{1/\gamma}(\bar{\theta}-\rho) + (\rho-\underline{\theta})} \\ x(\underline{m},\emptyset,\underline{\theta}) &= \frac{\alpha\rho\,(\bar{\theta}-\underline{\theta})}{(\bar{\theta}-\rho) + (\rho-\underline{\theta})\left(\frac{\bar{\pi}(\bar{\theta}-\rho)}{\underline{\pi}(\rho-\underline{\theta})}\right)^{1/\gamma}} \\ b &= \alpha\rho\left[\frac{1-\left(\frac{\underline{\pi}(\rho-\underline{\theta})}{\bar{\pi}(\bar{\theta}-\rho)}\right)^{1/\gamma}}{\left(\frac{\underline{\pi}(\rho-\underline{\theta})}{\bar{\pi}(\bar{\theta}-\rho)}\right)^{1/\gamma}(\bar{\theta}-\rho) + (\rho-\underline{\theta})}\right] - \alpha \\ \lambda &= \left[\frac{\underline{\pi}^{1/\gamma}(\bar{\theta}-\rho)^{\frac{\gamma-1}{\gamma}} + \bar{\pi}^{1/\gamma}(\rho-\underline{\theta})^{\frac{\gamma-1}{\gamma}}}{\alpha\rho\,(\bar{\theta}-\underline{\theta})^{\frac{\gamma-1}{\gamma}}}\right]^{\gamma} \\ \mu &= \frac{\bar{\pi}(\mathbb{E}(\theta)-\rho)}{\rho-\underline{\theta}}. \end{aligned} \tag{18}$$

B.2 Always audit contracts ($q = 1$) with no type-II errors ($\eta(\bar{\theta}) = 0$)

When audits occur with certainty following low type reports, and the audit signal correctly identifies high type entrepreneurs with certainty, then as in the private information case the incentive compatibility constraint becomes linear. Our problem can be expressed by the following Lagrangian

$$\begin{aligned}
\mathcal{L}_1 = {} & \bar{\pi}\, U(x(\bar{m}, \emptyset, \bar{\theta})) + \underline{\pi}(1 - \eta(\underline{\theta}))\, U(x(\underline{m}, \underline{\sigma}, \underline{\theta})) + \underline{\pi}\eta(\underline{\theta})\, U(x(\underline{m}, \bar{\sigma}, \underline{\theta})) \\
& + \lambda[(\alpha + b)(\mathbb{E}(\theta) - \rho - \underline{\pi}\kappa) + \alpha\rho - \bar{\pi}x(\bar{m}, \emptyset, \bar{\theta}) - \underline{\pi}(1 - \eta(\underline{\theta}))x(\underline{m}, \underline{\sigma}, \underline{\theta}) - \underline{\pi}\eta(\underline{\theta})x(\underline{m}, \bar{\sigma}, \underline{\theta})] \\
& + \mu \left[U(x(\bar{m}, \emptyset, \bar{\theta})) - U[x(\underline{m}, \bar{\sigma}, \underline{\theta}) + (\alpha + b)(\bar{\theta} - \underline{\theta})]\right] \qquad (19)
\end{aligned}$$

The first order conditions are

$$\begin{array}{lcrcl}
x(\bar{m}, \emptyset, \bar{\theta}) & : & 0 & = & \bar{\pi}U'(x(\bar{m}, \emptyset, \bar{\theta})) - \bar{\pi}\lambda + \mu\, U'(x(\bar{m}, \emptyset, \bar{\theta})) \\
x(\underline{m}, \underline{\sigma}, \underline{\theta}) & : & 0 & = & \underline{\pi}(1 - \eta(\underline{\theta}))U'(x(\underline{m}, \underline{\sigma}, \underline{\theta})) - \underline{\pi}(1 - \eta(\underline{\theta}))\lambda \\
x(\underline{m}, \bar{\sigma}, \underline{\theta}) & : & 0 & = & \underline{\pi}\eta(\underline{\theta})U'(x(\underline{m}, \bar{\sigma}, \underline{\theta})) - \underline{\pi}\eta(\underline{\theta})\lambda - \mu\, U'[x(\underline{m}, \bar{\sigma}, \underline{\theta}) + (\alpha + b)(\bar{\theta} - \underline{\theta})] \\
b & : & 0 & = & \lambda(\mathbb{E}(\theta) - \rho - \underline{\pi}\kappa) - \mu(\bar{\theta} - \underline{\theta})U'[x(\underline{m}, \bar{\sigma}, \underline{\theta}) + (\alpha + b)(\bar{\theta} - \underline{\theta})].
\end{array}$$

Substituting the incentive compatibility constraint into the first order conditions yields

$$\begin{aligned}
U'(x(\bar{m}, \emptyset, \bar{\theta})) &= \lambda \left(\frac{\bar{\pi}(\bar{\theta} - \underline{\theta}) - (\mathbb{E}(\theta) - \rho - \underline{\pi}\kappa)}{\bar{\pi}(\bar{\theta} - \underline{\theta})} \right) \\
U'(x(\underline{m}, \underline{\sigma}, \underline{\theta})) &= \lambda \\
U'(x(\underline{m}, \bar{\sigma}, \underline{\theta})) &= \lambda \left(\frac{\underline{\pi}\eta(\underline{\theta})(\bar{\theta} - \underline{\theta}) + \mathbb{E}(\theta) - \rho - \underline{\pi}\kappa}{\underline{\pi}\eta(\underline{\theta})(\bar{\theta} - \underline{\theta})} \right),
\end{aligned}$$

which we can combine with the intermediary's participation constraint and the utility function and solve for consumption allocations:

$$
\begin{aligned}
x(\bar{m}, \emptyset, \bar{\theta}) &= \alpha\rho\chi\left(\frac{1}{1-\zeta}\right)^{1/\gamma} \\
x(\underline{m}, \underline{\sigma}, \underline{\theta}) &= \alpha\rho\chi \\
x(\underline{m}, \bar{\sigma}, \underline{\theta}) &= \alpha\rho\chi\left(\frac{\underline{\pi}\eta(\underline{\theta})}{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})}\right)^{1/\gamma} \\
b &= \frac{\alpha\rho\chi}{\bar{\theta}-\underline{\theta}}\left(\frac{\zeta}{1-\zeta}\right)^{1/\gamma}\left(\frac{\bar{\pi}+\underline{\pi}\eta(\underline{\theta})}{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})}\right)^{1/\gamma} - \alpha \\
\lambda &= (\alpha\rho\chi)^{-\gamma} \\
\mu &= \frac{\bar{\pi}\zeta}{1-\zeta}.
\end{aligned}
\tag{20}
$$

where

$$
\chi = \frac{1}{\bar{\pi}\,(1-\zeta)^{\frac{\gamma-1}{\gamma}} + \underline{\pi}\eta(\underline{\theta})\left(\frac{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})}{\underline{\pi}\eta(\underline{\theta})}\right)^{\frac{\gamma-1}{\gamma}} + \underline{\pi}(1-\eta(\underline{\theta}))}, \qquad \text{and}
$$

$$
\zeta = \frac{\mathbb{E}(\theta) - \rho - \underline{\pi}\kappa}{\bar{\pi}(\bar{\theta}-\underline{\theta})}.
$$

B.3 Proof of proposition 5

Proof. We can write the first order condition for q as follows:

$$
\begin{aligned}
\mathcal{L}_q = {} & (1-\eta(\underline{\theta}))U(x(\underline{m}, \underline{\sigma}, \underline{\theta})) + \eta(\underline{\theta})U(x(\underline{m}, \bar{\sigma}, \underline{\theta})) - U(x(\underline{m}, \emptyset, \underline{\theta})) \\
& - \lambda[(\alpha+b)\pi\kappa + (1-\eta(\underline{\theta}))x(\underline{m}, \underline{\sigma}, \underline{\theta}) + \eta(\underline{\theta})x(\underline{m}, \bar{\sigma}, \underline{\theta}) - x(\underline{m}, \emptyset, \underline{\theta})] \\
& + \mu[U(x(\underline{m}, \emptyset, \underline{\theta}) + (\alpha+b)(\bar{\theta}-\underline{\theta})) - U(x(\underline{m}, \bar{\sigma}, \underline{\theta}) + (\alpha+b)(\bar{\theta}-\underline{\theta}))] + \nu_0 - \nu_1.
\end{aligned}
\tag{21}
$$

In order to solve for Kuhn-Tucker multiplier ν_1, we need to solve for shadow allocations which are consumed with probability zero: $x(\underline{m}, \emptyset, \underline{\theta})$ in the case of standard debt.

In the limit as $q \to 1^-$, the first order condition for $x(\underline{m}, \emptyset, \underline{\theta})$ must hold, even though this allocation is consumed with vanishing probability. The first order condition for $x(\underline{m}, \emptyset, \underline{\theta})$ is:

$$x(\underline{m}, \emptyset, \underline{\theta}): \quad 0 = \underline{\pi} U'(x(\underline{m}, \emptyset, \underline{\theta})) - \underline{\pi}\lambda - \mu\, U'[x(\underline{m}, \emptyset, \underline{\theta}) + (\alpha + b)(\bar{\theta} - \underline{\theta})].$$

Under logarithmic utility, $U'(x) = 1/x$. We can solve for $x(\underline{m}, \emptyset, \underline{\theta})$ using the solutions obtained in (20):

$$\begin{aligned} x(\underline{m}, \emptyset, \underline{\theta}) = & -\frac{\alpha\rho}{2}\left[\frac{1}{\underline{\pi}}\frac{\zeta}{1-\zeta} - \frac{\underline{\pi}\eta(\underline{\theta})}{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})}\right] \\ & + \frac{\alpha\rho}{2}\sqrt{\left[\frac{1}{\underline{\pi}}\frac{\zeta}{1-\zeta} - \frac{\underline{\pi}\eta(\underline{\theta})}{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})}\right]^2 + 4\frac{\zeta}{1-\zeta}\left(\frac{\bar{\pi} + \underline{\pi}\eta(\underline{\theta})}{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})}\right)}. \end{aligned} \tag{22}$$

Noting that $\nu_0 = 0$ by complimentary slackness, substitute (22) and (20) into (21) to express the Kuhn-Tucker multiplier ν_1 as a closed-form expression in terms of parameters. ■

C PROOF OF PROPOSITION 6

Proof. Consider the case where the entrepreneur enjoys consumption with log utility, type–II errors occur with zero probability ($\eta(\bar{\theta}) = 0$) and type-I errors occur with positive but very low probability, $\eta(\underline{\theta}) \to 0^+$. By proposition 4, consumption following overturned low type reports also tends toward zero ($\lim_{\eta(\underline{\theta})\to 0^+} x(\underline{m}, \bar{\sigma}, \underline{\theta}) = 0$), and by l'Hôpital's rule, the contribution to ex ante expected utility of the entrepreneur from consumption following overturned reports also tends toward zero, ($\lim_{\eta(\underline{\theta})\to 0^+} \underline{\pi}\eta(\underline{\theta})U(x(\underline{m}, \bar{\sigma}, \underline{\theta})) = 0$). When type-I errors are extremely rare, errors have little effect on the ex ante welfare of entrepreneurs but still limit repayments drawn from high type entrepreneurs.

For tractability, we consider contracts under the following assumptions: First, type-I errors will occur with very low (positive) probability. Second, utility will be logarithmic over consumption. Third, the two income states will occur with equal probability.

Let y_1 (y_0) be the efficient value of choice variable y in the always audit (private information) contract. Substituting $U(x) = \log x$ and $\underline{\pi} = \bar{\pi} = 1/2$ into the solutions from Appendix B, and taking the limit as $\eta(\underline{\theta}) \to 0^+$, we obtain the following solutions:

$$\frac{\alpha+b_1}{\alpha}=\frac{1}{2}\frac{\rho}{\rho-\underline{\theta}+\underline{\pi}\kappa},\qquad x_1(\bar{m},\emptyset,\bar{\theta})=\alpha\rho\frac{1}{2}\frac{\bar{\theta}-\underline{\theta}}{\rho-\underline{\theta}+\underline{\pi}\kappa},\qquad x_1(\underline{m},\underline{\sigma},\underline{\theta})=\alpha\rho$$

$$x_1(\underline{m},\emptyset,\underline{\theta})=\alpha\rho\frac{-(\mathbb{E}(\theta)-\rho-\underline{\pi}\kappa)+\sqrt{(\mathbb{E}(\theta)-\rho-\underline{\pi}\kappa)^2+\frac{1}{2}(\bar{\theta}-\underline{\theta})(\rho-\underline{\theta}+\underline{\pi}\kappa)}}{\rho-\underline{\theta}+\underline{\pi}\kappa}$$

$$x_1(\underline{m},\bar{\sigma},\underline{\theta})=0,\qquad \lambda_1=\frac{1}{\alpha\rho},\qquad \mu_1=\frac{1}{2}\frac{\mathbb{E}(\theta)-\rho-\underline{\pi}\kappa}{\rho-\underline{\theta}+\underline{\pi}\kappa}. \tag{23}$$

$$\frac{\alpha+b_0}{\alpha}=\frac{\rho(\mathbb{E}(\theta)-\rho)}{(\bar{\theta}-\rho)(\rho-\underline{\theta})},\qquad x_0(\bar{m};\emptyset;\bar{\theta})=\alpha\rho\frac{1}{2}\frac{\bar{\theta}-\underline{\theta}}{\rho-\underline{\theta}},\qquad x_0(\underline{m},\underline{\sigma},\underline{\theta})=\alpha\rho$$

$$x_0(\underline{m},\emptyset,\underline{\theta})=\alpha\rho\,\frac{1}{2}\frac{\bar{\theta}-\underline{\theta}}{\bar{\theta}-\rho},\qquad x_0(\underline{m},\bar{\sigma},\underline{\theta})=0,\qquad \lambda_0=\frac{1}{\alpha\rho},\qquad \mu_0=\frac{1}{2}\frac{\mathbb{E}(\theta)-\rho}{\rho-\underline{\theta}}. \tag{24}$$

The bang-bang result occurs when efficient contracts 'jump' between private information and standard debt contracts, where both are local maxima. To focus on these bang-bang results, we first set κ such that the always audit and private information contracts provide equal expected utility ($\mathbb{E}U(x_1)=\mathbb{E}U(x_0)$). Solving for κ yields

$$\underline{\pi}\kappa=2\,\frac{(\rho-\underline{\theta})(\mathbb{E}(\theta)-\rho)}{\bar{\theta}-\underline{\theta}} \tag{25}$$

It is useful to define two new parameters, one representing the excess return in good states and the other the shortfall in bad states. Thus, let $\bar{\phi},\underline{\phi}\in\mathbb{R}^+$, where $\bar{\phi}=\bar{\theta}-\rho$, and $\underline{\phi}=\rho-\underline{\theta}$. All else equal, the entrepreneur would prefer a project with large $\bar{\phi}$, and small $\underline{\phi}$. Note that assumptions 1 and 2 require that $\underline{\phi}\in(0,\bar{\phi})$. Substituting equation 25 into the solutions for the always standard debt contract (23), we can re-write allocations as follows:

$$b_1=\frac{\alpha\rho}{4}:\frac{\bar{\phi}+\underline{\phi}}{\bar{\phi}\underline{\phi}}-\alpha,\qquad x_1(\bar{m},\emptyset,\bar{\theta})=\frac{\alpha\rho}{4}\frac{(\bar{\phi}+\underline{\phi})^2}{\bar{\phi}\underline{\phi}},$$

$$x_1(\underline{m},\emptyset,\underline{\theta})=\frac{\alpha\rho}{4}\left[\frac{-(\bar{\phi}-\underline{\phi})^2+\sqrt{(\bar{\phi}-\underline{\phi})^4+4\bar{\phi}\underline{\phi}(\bar{\phi}+\underline{\phi})^2}}{\bar{\phi}\underline{\phi}}\right],\qquad \mu_1=\frac{1}{8}\,\frac{(\bar{\phi}-\underline{\phi})^2}{\bar{\phi}\underline{\phi}} \tag{26}$$

Now, consider the trade-off characterised by the first order condition for auditing q at the always audit contract. After substituting equations 25 and

26 into the first order condition for q (equation 21), we obtain

$$\mathcal{L}_q = \frac{1}{2}\log\left[\frac{4\bar{\phi}\underline{\phi}}{-(\bar{\phi}-\underline{\phi})^2+\sqrt{(\bar{\phi}-\underline{\phi})^4+4\bar{\phi}\underline{\phi}(\bar{\phi}+\underline{\phi})^2}}\right]$$
$$-\frac{1}{2}\left[1-\frac{-(\bar{\phi}-\underline{\phi})^2+\sqrt{(\bar{\phi}-\underline{\phi})^4+4\bar{\phi}\underline{\phi}(\bar{\phi}+\underline{\phi})^2}}{4\bar{\phi}\underline{\phi}}\right]-\frac{1}{4}\frac{(\bar{\phi}-\underline{\phi})}{\bar{\phi}}$$
$$+\frac{1}{8}\frac{(\bar{\phi}-\underline{\phi})^2}{\bar{\phi}\underline{\phi}}\log\left(1+\frac{-(\bar{\phi}-\underline{\phi})^2+\sqrt{(\bar{\phi}-\underline{\phi})^4+4\bar{\phi}\underline{\phi}(\bar{\phi}+\underline{\phi})^2}}{(\bar{\phi}+\underline{\phi})^2}\right)-\nu_1. \tag{27}$$

The first term on the right hand side of equation 27 is the welfare gain attained through auditing by verifying low type agents' reports. In this example, agents clearly prefer to be audited, $x(\underline{m},\underline{\sigma},\underline{\theta}) > x(\underline{m},\emptyset,\underline{\theta})$ for all values of $\bar{\phi},\underline{\phi}$. The second term captures the resource cost of this increase in consumption for low type agents whose reports are verified, and the third term represents the extra resource costs expended by the intermediary in conducting more audits. The fourth term captures welfare gains attained by relaxing the incentive compatibility constraint: auditing with a higher probability directly increases the likelihood that misreporting high type entrepreneurs will be punished. The final term on the right hand side is the Lagrange multiplier capturing the shadow cost of the natural upper bound of one attached to the audit probability.

Let $\underline{\phi}\to 0^+$. By l'Hôpital's rule,

$$\lim_{\underline{\phi}\to 0^+}\frac{-(\bar{\phi}-\underline{\phi})^2+\sqrt{(\bar{\phi}-\underline{\phi})^4+4\bar{\phi}\underline{\phi}(\bar{\phi}+\underline{\phi})^2}}{4\bar{\phi}\underline{\phi}}=\frac{1}{2}, \qquad \text{and}$$

$$\lim_{\underline{\phi}\to 0^+}\frac{1}{\underline{\phi}}\log\left(1+\frac{-(\bar{\phi}-\underline{\phi})^2+\sqrt{(\bar{\phi}-\underline{\phi})^4+4\bar{\phi}\underline{\phi}(\bar{\phi}+\underline{\phi})^2}}{(\bar{\phi}+\underline{\phi})^2}\right)=\frac{2}{\bar{\phi}}.$$

Substituting these results into 27 while retaining the same ordering of terms yields

$$\lim_{\underline{\phi}\to 0^+}\mathcal{L}_q(q=1)=\frac{1}{2}\log 2-\frac{1}{4}-\frac{1}{4}+\frac{1}{4}-\nu_1. \tag{28}$$

The Kuhn-Tucker multiplier ν_1 is positive. At the margin, the benefits of additional audits outweigh the costs. The first two terms show that the utility benefits accruing to low type entrepreneurs from verification of their reports exceeds the resource cost associated with awarding more low type entrepreneurs with the post-verification consumption bundle. The resource cost of audits is the product of the Lagrange multiplier on the resource constraint, total assets devoted to the project and audit costs. Here, as the downside shortfall $\underline{\phi}$ approaches zero, borrowing and assets devoted to the project are unbounded above. When downside risk is low, the benefits of auditing are small, and indeed the audit cost which equates the expected welfare of always audit and private information contracts is vanishing $\underline{\pi}\kappa \to 0^+$. The resource cost of the marginal audit is 1/4, which in this case is equal and opposite to the benefit attained from the marginal audit by relaxing the incentive compatibility constraint.

For the same case, letting $\underline{\phi} \to 0^+$, now consider the corresponding private information contract. The first order condition for q can be written as follows:

$$\mathcal{L}_q = \frac{1}{2}\log\frac{2\bar{\phi}}{\bar{\phi}+\underline{\phi}} - \frac{1}{2}\left[1 - \frac{1}{2}\frac{\bar{\phi}+\underline{\phi}}{\bar{\phi}}\right] - \frac{1}{2}\frac{(\bar{\phi}-\underline{\phi})^2}{\bar{\phi}(\bar{\phi}+\underline{\phi})} + \frac{1}{4}\frac{\bar{\phi}-\underline{\phi}}{\underline{\phi}}\log\left(\frac{\underline{\phi}}{\bar{\phi}-\underline{\phi}}+1\right) + \nu_0.$$

Taking the limit as $\underline{\phi} \to 0^+$ yields

$$\lim_{\underline{\phi}\to 0^+} \mathcal{L}_q(q=0) = \frac{1}{2}\log 2 - \frac{1}{4} - \frac{1}{2} + \frac{1}{4} + \nu_0, \tag{29}$$

The first term captures the direct welfare benefit from verifying entrepreneur reports, and providing them with the consumption bundle $x_0(\underline{m}, \underline{\sigma}, \underline{\theta}) > x_0(\underline{m}, \emptyset, \underline{\theta})$. This benefit, and the resource cost associated with it and captured in the second term, are identical to those in the always audit contract (28). This is due to the fact that these consumption bundles are identical in both contracts for this limiting case: low type agents whose reports are verified $(x(\underline{m}, \underline{\sigma}, \underline{\theta}))$ consume $\alpha\rho$ and low type agents whose reports are unverified $(x(\underline{m}, \emptyset, \underline{\theta}))$ consume $\alpha\rho/2$ in both contracts. As in the always audit contract (28), the fourth term capturing the relaxation in the incentive compatibility constraint is equal to $1/4$.

The third term, capturing the cost of the marginal audit, is greater in magnitude than under the always audit contract. Audit costs as a fraction of assets employed in the project are constant across contracts by assumption, yet leverage in the private information contract is greater than in the always audit contract. We can see this by taking the limit of the ratio of assets devoted to the project in the two contracts: $\lim_{\underline{\phi}\to 0} \dfrac{\alpha + b_0}{\alpha + b_1} = 2$.

■

D STANDARD DEBT CONTRACTS WITH TYPE-II ERRORS

Assume that the probability of Type-II errors $\eta(\bar{\theta})$ following the audit of a high type entrepreneur is low, and that the optimal contract is standard debt ($q^* = 1$). We can find approximate closed form solutions to optimal contracts using a first order Taylor expansion of the Incentive Compatibility Constraint around $\eta(\bar{\theta}) = 0$. Denote the efficient contract at $\eta(\bar{\theta}) = 0$ with all other parameters constant by Γ_0, with associated public actions and allocations labelled $b_0, q_0 = 1, x_0, z_0$.

$$\begin{aligned} U(x_0(\bar{m}, \emptyset, \bar{\theta})) \geq\ & U(x_0(\underline{m}, \bar{\sigma}, \bar{\theta})) + U'(x_0(\underline{m}, \bar{\sigma}, \bar{\theta}))[x(\underline{m}, \bar{\sigma}, \bar{\theta}) - x_0(\underline{m}, \bar{\sigma}, \bar{\theta})] \\ & - U'(x_0(\bar{m}, \emptyset, \bar{\theta}))[x(\bar{m}, \emptyset, \bar{\theta}) - x_0(\bar{m}, \emptyset, \bar{\theta})] \\ & + \eta(\bar{\theta})[U(x_0(\underline{m}, \underline{\sigma}, \bar{\theta})) - U(x_0(\underline{m}, \bar{\sigma}, \bar{\theta}))] \end{aligned}$$

Note that $x_0(\bar{m}, \emptyset, \bar{\theta}) = x_0(\underline{m}, \bar{\sigma}, \bar{\theta})$, which combined with the budget constraints allows us to simplify the above expression as follows:

$$z(\underline{m}, \bar{\sigma}) - z(\bar{m}, \emptyset) \geq \frac{\eta(\bar{\theta})}{U'(x_0(\underline{m}, \bar{\sigma}, \bar{\theta}))}[U(x_0(\underline{m}, \underline{\sigma}, \bar{\theta})) - U(x_0(\underline{m}, \bar{\sigma}, \bar{\theta}))]. \tag{30}$$

When utility is log, we can solve directly using Proposition 4. After rearranging and simplifying, our first order approximation of the ICC can be written as a linear expression in terms of contracted repayments:

$$z(\underline{m}, \bar{\sigma}) - z(\bar{m}, \emptyset) \geq \eta(\bar{\theta})\ \frac{\alpha\rho}{1-\zeta} \log\left[1 + \frac{\bar{\pi}\zeta(1-\bar{\pi}\zeta)}{\bar{\pi}\zeta + \underline{\pi}\eta(\underline{\theta})}\right] \tag{31}$$

This linear approximation to the ICC permits closed form approximations to allocations and leverage for efficient contracts, which are not presented here.

One interpretation of equation 31 is that it specifies a a small non-refundable fee paid by all entrepreneurs who declare a low type return. Entrepreneurs whose reports are overturned by the audit signal would be required to repay the full contracted repayment $z(\bar{m}, \emptyset)$ in addition to the small fee. When $\eta(\bar{\theta})$ is small, this fee is negligible.

E Figures

Figure 1: Efficient contracts, project risk and audit costs. $(U(x) = \sqrt{x}, \rho = 1, \mathbb{E}(\theta) = 1.2, \bar{\pi} = 0.9, \eta(\underline{\theta}) = 0.01)$.

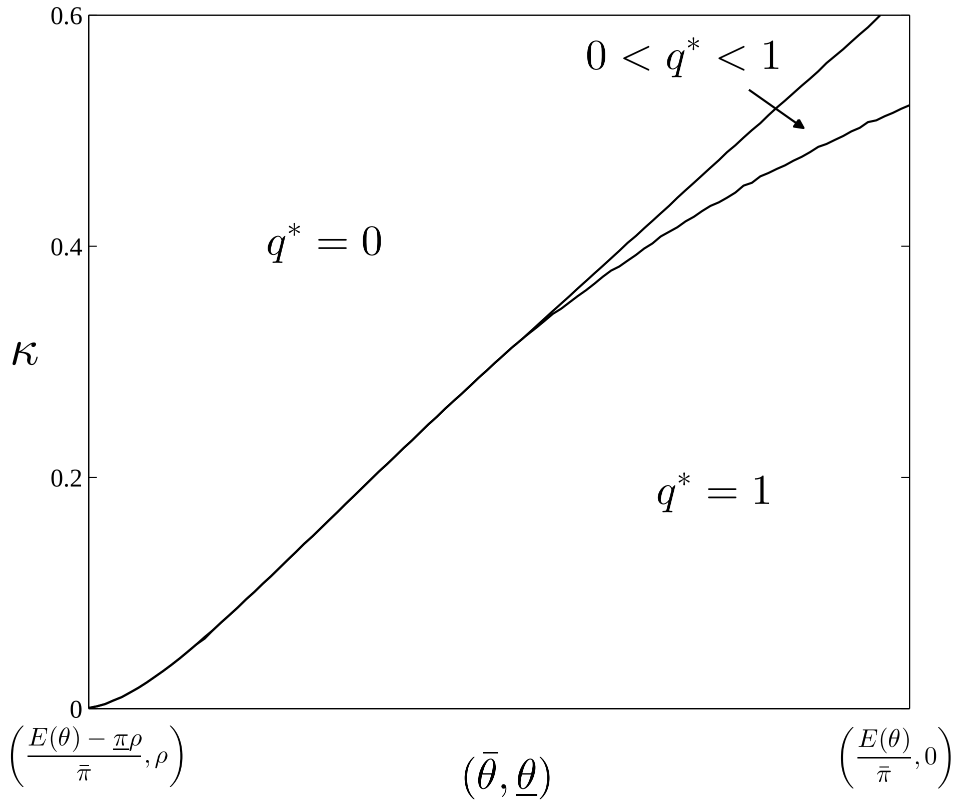

Figure 2: The determination of optimal contracts when there are multiple local maxima. $(U(x) = \log x, \rho = 1, \mathbb{E}(\theta) = 1.2, \bar{\pi} = 0.9, (\bar{\theta} - \underline{\theta}) = 0.3, \eta(\underline{\theta}) = 10^{-4}, \alpha = 1, \kappa \approx 0.18)$

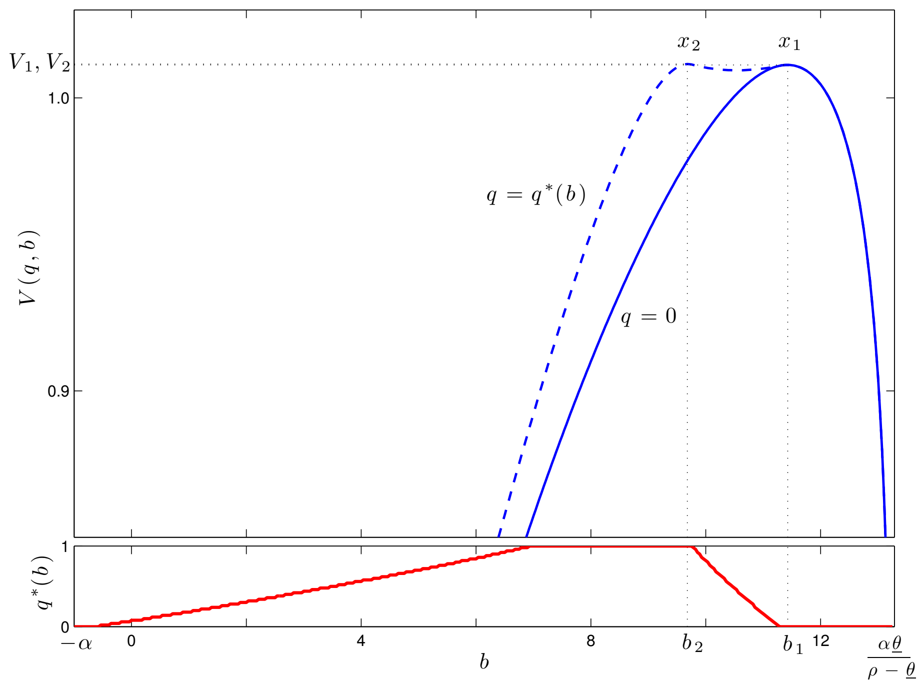

Figure 3: A Four-state example. ($\Theta = (0.9, 1.0, 1.1, 1.2), \pi(\Theta) = (1/8, 3/8, 3/8, 1/8), \rho = 1, \alpha = 1, b = 5, \kappa = 0.08, U(x) = x^{1-\gamma}/(1-\gamma), \gamma = 9/10$). Imperfect audits case marked by $\times$, with $P(\sigma = \theta_i|\theta = \theta_j) = 10^{-|i-j|}$ if and only if $i \neq j$. Perfect audits case marked by $+$, with $P(\sigma = \theta_i|\theta = \theta_j) = 0$ if and only if $i \neq j$.