Expectations Traps and Coordination Failures with Discretionary Policymaking*

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Abstract

Discretionary policymakers cannot manage private-sector expectations and cannot coordinate the actions of future policymakers. As a consequence, expectations traps and coordination failures can occur and multiple equilibria can arise. To utilize the explanatory power of models with multiple equilibria it is first necessary to understand how an economy arrives to a particular equilibrium. In this paper we employ notions of learnability and self-enforceability to motivate and identify equilibria of particular interest. Central among these criteria are whether the equilibrium is learnable by private agents and jointly learnable by private agents and the policymaker. We use two New Keynesian policy models to identify the strategic interactions that give rise to multiple equilibria and to illustrate our methods for identifying equilibria of interest. Importantly, unless the Pareto-preferred equilibrium is learnable by private agents, we find little reason to expect coordination on that equilibrium.

Key Words: Discretionary policymaking, multiple equilibria, coordination, equilibrium selection.

JEL References: E52, E61, C62, C73

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1 Introduction

Discretionary policymakers can fall foul of expectations traps and coordination failures. When private agents are forward-looking their expectations, shaped by anticipations about future policy, influence how policy today is conducted. The discretionary policymaker's Achilles heel is that when formulating policy it is unable to manage private sector expectations, and this inability, inherent to time-consistent policymaking, leaves ajar the door to multiple equilibria. When expectations cannot be managed, private agents can form expectations that, although unwelcome from the policymaker's perspective, lead private agents to react in a manner that traps the policymaker into implementing a policy that validates those expectations. The trap is closed when a policy that renders those unwelcome expectations without foundation is more costly and hence less attractive to the discretionary policymaker than a policy that accommodates them.

The fact that multiple equilibria produced by the policymaker's inability to manage private sector expectations can beset discretionary control problems is troublesome, yet hugely important. Troublesome, because efforts to solve or mitigate the time-consistency problem associated with optimal policymaking rely invariably on there being a unique discretionary equilibrium. A Rogoff-style (Rogoff, 1985) approach of delegating objectives to a discretionary policymaker (as per Jensen and McCallum (2002) and Walsh (2003), among others) is unlikely to be successful unless it also solves the coordination problem. Similarly, to the extent that an optimal contract (Walsh, 1995) can successfully overcome the time-consistency problem, it too should address the coordination problem. Important, because it means that discretionary policy behavior can be considerably richer and more varied than is commonly appreciated, with switches among equilibria becoming a potential source of economic volatility. Moreover, because the mechanisms that produce multiple equilibria involve strategic interactions between agents over time, they are not precluded by linear constraints and quadratic objectives. As a consequence, much research analyzing discretionary policymaking since Kydland and Prescott (1977) may have inadvertently considered only one of several equilibria, potentially overlooking essential aspects of discretionary policy behavior.

It is not unusual for economies to transition between periods of high and low inflation, a phenomenon that expectations traps have the potential to explain (Albanesi, Chari, and Christiano, 2003). Similarly, transitions from one equilibrium to another offers an explanation for policy regime changes, like those analyzed by Davig and Leeper (2006). Accordingly, an explanation for the change in U.S. inflation behavior between the 1970s and the 1980s could be that Volcker's appointment to Federal Reserve Chairman served to coordinate expectations and behavior, switching the economy from one discretionary equilibrium to another. However, in order to utilize the explanatory power of multiple equilibria it is necessary to first consider how an economy arrives at a particular equilibrium. In the words of Benabib and Farmer (1999, pp. 438) "in any model with multiple equilibria one must address the issue of how an equilibrium comes about".

In this paper we consider the issue of how the agents residing in a model may coordinate on an equilibrium. We suggest two approaches, based on different interpretations of the interactions occurring among the economic agents in the model. First, following Backus and Driffill (1986) and Currie and Levine (1993), we describe the discretionary policy problem as a *control problem.* There is an infinitely-lived policymaker who faces a continuum of atomistic agents, and all agents seek to rationalize a discretionary equilibrium based on a simple and relevant behavioral model.

We, therefore, consider learning as a coordinating mechanism for equilibrium selection (Evans, 1986), drawing on the large literature that employs learning to analyze coordination in rational expectations models (Guesnerie and Woodford, 1992; Evans and Guesnerie, 1993, 2003, 2005; Evans and Honkapohja, 2001). Allowing private agents and/or the policymaker to be learning, we develop expectational stability conditions whose satisfaction determines whether private agents and/or the policymaker might reasonably learn and coordinate on a particular equilibrium. Among these stability conditions, we show that the key conditions are those indicating whether an equilibrium is learnable by private agents in isolation and by private agents and the policymaker jointly.

Second, drawing on Cohen and Michel (1988) and Oudiz and Sachs (1985), we reinterpret the control problem as a *dynamic game* between policymakers at different points in time. Feedback equilibria of the discretionary control problem correspond to Markov-perfect Nash equilibria of the dynamic game. We show how strategic interactions among current and future policymakers, operating through endogenous state variables and private sector expectations, leads to a form of strategic complementarity (Cooper and John, 1988) and makes expectations traps and coordination failures possible. Although unilateral deviations from a Nash equilibrium are not beneficial, several policymakers can form a coalition. We consider whether the potential for such coalitions to form might effectively rule out some equilibria (Bernheim, Peleg, and M.Whinston, 1987). Pursuing this idea, we examine whether the Nash equilibria we obtain are self-enforceable.

We introduce these coordination mechanisms by means of two models with multiple discretionary equilibria. The first model is a version of the sticky price model with government debt adapted from Leeper (1991) by Blake and Kirsanova (2012). The second model is a sticky price New Keynesian model in the spirit of Woodford (2003, Ch.5), but with inflation indexation. In each model, the task confronting the policymaker is to stabilize inflation without impacting unduly the real economy. Inflation, in these models, is determined by the expected path of real marginal costs, so the policy challenge is to generate an appropriate path for real marginal costs. Since inflation depends on the entire expected path for real marginal costs while the discretionary policymaker can choose only today's policy, the policy chosen today depends necessarily on expected future policy. At the same time, the decisions that future policymakers make depend materially on the economic circumstances that they find themselves in, and hence on the choices that previous policymakers have made. This interaction between policymakers over time produces coordination failure and leads to multiple equilibria.

The model with government debt is relatively simple, allowing us to introduce the coordination mechanisms and derive most of our results in analytical form. In contrast, the second model is a version of the standard Dynamic Stochastic General Equilibrium (DSGE) model that resides at the core of many New Keynesian models, such as those developed by Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2007). As such, it is much more complex and can only be solved numerically, but it allows us to demonstrate how the coordination mechanisms work in empirically relevant, but analytically less tractable, models.

The remainder of the paper is structured as follows. In Section 2, we motivate and describe mechanisms by which agents may coordinate on an equilibrium and apply them to the government-debt model, obtaining all results in analytical form. In Section 3 we apply our coordination mechanisms to the DSGE model. Section 4 concludes. We delegate the formal treatment of the coordination mechanisms in the linear-quadratic class of models to Appendix A.

2 A simple New Keynesian model with government debt

The model summarized below draws on Leeper (1991) and was used by Blake and Kirsanova (2012) to demonstrate the existence of multiple discretionary equilibria in linear-quadratic models. We use it here to illustrate potential coordination mechanisms because its simplicity allows us to demonstrate all of our results in analytical form.

The economy is populated by a representative household, by a unit-continuum of monopolistically competitive firms, and by a single large government that conducts separately monetary policy and fiscal policy. Fiscal policy is conducted via a mechanistic rule that relates the income tax rate to the stock of real government debt. Monetary policy, in contrast, is conducted by choosing the nominal interest rate on a one-period nominal bond optimally, but under discretion. Importantly, when formulating monetary policy the central bank takes the fiscal rule into account. Monopolistically competitive firms produce according to a production function that depends only on labor, and these goods are combined via a Dixit and Stiglitz (1977) technology to produce aggregate output which is allocated to either private consumption or government spending. Households choose their consumption and leisure and can transfer income over time through their holdings of government bonds. The government issues debt period-by-period in order to pay the principle and interest on its existing debt and to fund any budget deficit. Firms set prices subject to a Calvo (1983) nominal price rigidity and aggregation across prices leads to a New Keynesian Phillips curve relating inflation to the expected future inflation, real marginal costs, and a serially correlated markup shock.

The model's complete derivation and first-order approximation can be found in Blake and Kirsanova (2012). However, when approximated about an efficient zero-inflation nonstochastic steady state, it is described by

$$\begin{aligned} b_{t+1} &= \rho b_t - \eta c_t, && (1)\\ \pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \lambda c_t + \nu b_t + u_t, && (2)\\ u_{t+1} &= \rho_u u_t + \epsilon_{t+1}, && (3) \end{aligned}$$

where b_t represents the ratio of real government debt to output, c_t represents real consumption, π_t represents inflation, u_t represents a markup, or cost-push, shock, and the innovation, ϵ_t is distributed $i.i.d.[0, \sigma_\epsilon^2]$. While $\beta \in (0,1)$ denotes the discount factor and $\rho_u \in (0,1)$ denotes the persistence of the markup shock, $\rho \in (0,1)$, $\eta \in (0, \beta^{-1})$, $\lambda \in (0,\infty)$, and $\nu \in (0,\infty)$ are convolutions of behavioral parameters—preference and technology parameters—and of the fiscal response of the income-tax-rate to debt.

The policymaker's intertemporal welfare criterion is described by the quadratic loss function

$$L_t = \mathbb{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\pi_s^2 + \alpha c_s^2 \right), \tag{4}$$

where $\alpha \in (0,\infty)$ is also a convolution of the model's behavioral parameters, derived under the assumption that the monopolistic distortion is offset by a labor subsidy, financed lump-sum, see Woodford (2003).

2.1 Discretionary equilibria

The description of the aggregate economy, equations (1)—(3), is known by the policymaker and is taken into account when he formulates the optimal policy. Today's policymaker determines their best action, knowing that future policymakers have the freedom to change policy but will apply the same decision process. With quadratic objectives and linear constraints, in a time-consistent equilibrium the decision rules for consumption (the policymaker) and inflation (private sector) will be linear functions of the current state

$$
\begin{aligned}
c_t &= c_u u_t + c_b b_t, && (5)\\
\pi_t &= \pi_u u_t + \pi_b b_t && (6)
\end{aligned}
$$

and the policymaker's value function can be written as

$$V(u_t, b_t) = S_{uu} u_t^2 + 2S_{ub} u_t b_t + S_{bb} b_t^2. \tag{7}$$

Combining equation (6) with the New Keynesian Phillips curve, equation (2) yields the private-sector reaction function

$$\pi_t = (\beta \pi_u \rho + 1) u_t + (\beta \pi_b \rho + \nu) b_t + (\lambda - \beta \pi_b \eta) c_t. \tag{8}$$

Optimal discretionary policy is characterized by the solutions to the Bellman equation

$$V(u_t, b_t) = \min_{c_t} \left[((\beta \rho \pi_u + 1) u_t + (\beta \rho \pi_b + \nu) b_t + (\lambda - \beta \eta \pi_b) c_t)^2 + \alpha c_t^2 + \beta V(u_{t+1}, b_{t+1}) \right],$$

with the constraints given by equations (1) and (3).

Given equation (7), the policymaker's optimal discretionary policy can be written in form of equation (5) with the response coefficients

$$
\begin{aligned}
c_u &= -\frac{((\lambda - \beta \eta \pi_b)(\beta \rho \pi_u + 1) - \eta \beta \rho S_{ub})}{\beta \eta^2 S_{bb} + (\lambda - \beta \eta \pi_b)^2 + \alpha} && (9)\\
c_b &= -\frac{((\lambda - \beta \eta \pi_b)(\beta \rho \pi_b + \nu) - \eta \beta \rho S_{bb})}{\beta \eta^2 S_{bb} + (\lambda - \beta \eta \pi_b)^2 + \alpha}. && (10)
\end{aligned}
$$

Accordingly, the value-function coefficients satisfy

$$
\begin{aligned}
S_{uu} &= ((\beta \pi_u \rho + 1) + (\lambda - \beta \pi_b \eta) c_u)^2 + \alpha c_u^2 + \beta \left(\rho^2 S_{uu} - 2\rho \eta S_{ub} c_u + \eta^2 S_{bb} c_u^2\right), && (11)\\
S_{ub} &= ((\beta \pi_u \rho + 1) + (\lambda - \beta \pi_b \eta) c_u)((\beta \pi_b \rho + \nu) + (\lambda - \beta \pi_b \eta) c_b) + \alpha c_u c_b && (12)\\
&\quad + \beta S_{ub} \rho (\rho - \eta c_b) - \beta S_{bb} \eta c_u (\rho - \eta c_b), \\
S_{bb} &= ((\beta \pi_b \rho + \nu) + (\lambda - \beta \pi_b \eta) c_b)^2 + \beta S_{bb} (\rho - \eta c_b)^2 + \alpha c_b^2, && (13)
\end{aligned}
$$

and the decision rule for inflation, equation (6), has the response coefficients

$$
\begin{aligned}
\pi_u &= \beta \rho \pi_u + 1 + (\lambda - \beta \eta \pi_b) c_u, && (14)\\
\pi_b &= \beta \rho \pi_b + \nu + (\lambda - \beta \eta \pi_b) c_b. && (15)
\end{aligned}
$$

For this decision problem, any set of coefficients $\{c_u, c_b, \pi_u, \pi_b, S_{uu}, S_{ub}, S_{bb}\}$ that satisfies equations (9)—(15) represents a discretionary equilibrium. Although it is not obvious, this model

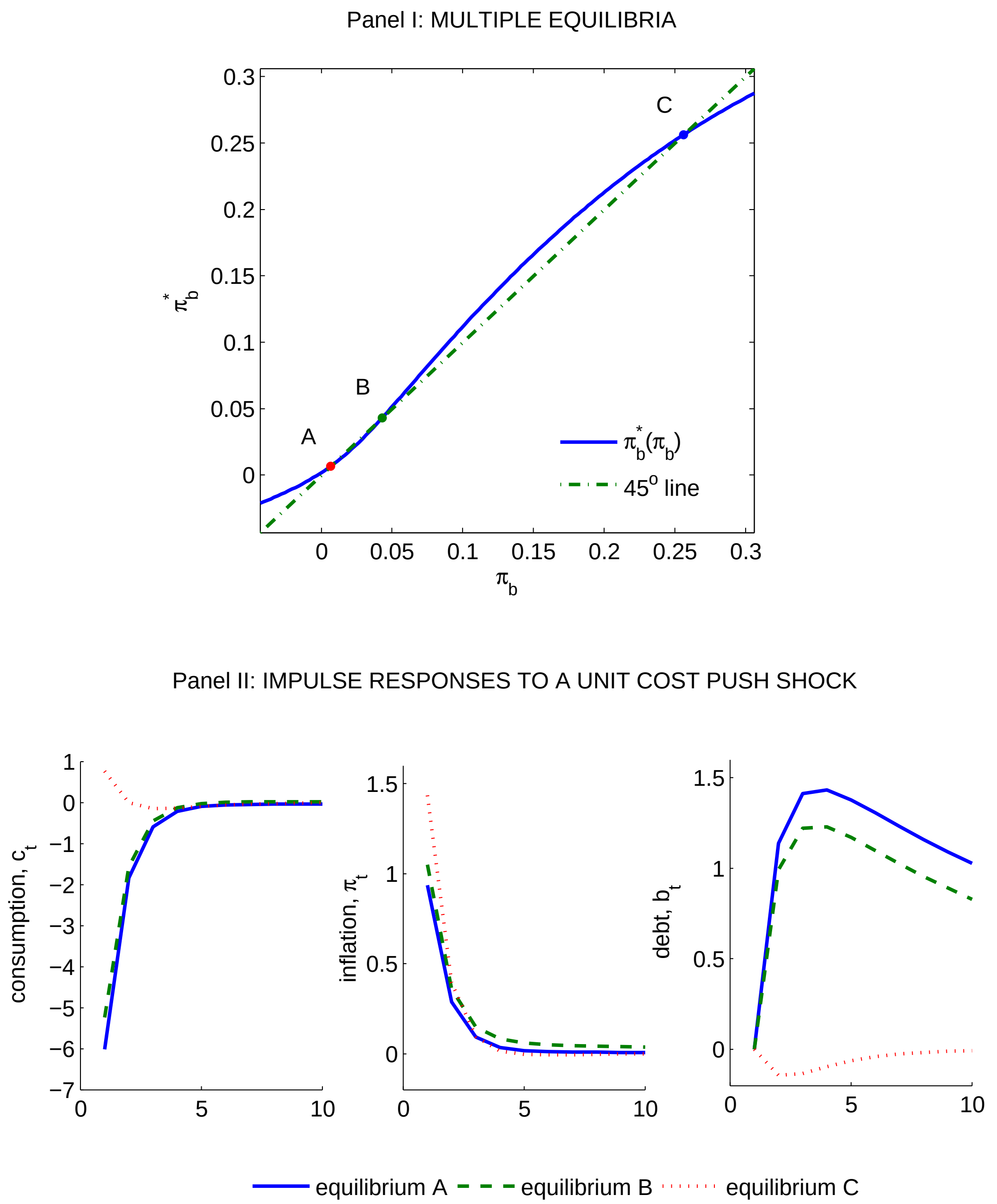

Figure 1: New Keynesian model with government debt

Eqm.	Policy Reaction	Private sector Reaction	Loss matrix	Speed of adjustment	Unconditional loss
	$\begin{bmatrix} c_\eta & c_b \end{bmatrix}$	$\begin{bmatrix} \pi_\eta & \pi_b \end{bmatrix}$	$\begin{bmatrix} S_{\eta\eta} & S_{\eta b} \\ S_{\eta b} & S_{bb} \end{bmatrix}$	b_b	$\mathbb{E}[L]$
A	$\begin{bmatrix} -6.014 & -0.034 \end{bmatrix}$	$\begin{bmatrix} 0.935 & 0.007 \end{bmatrix}$	$\begin{bmatrix} 1.316 & 0.012 \\ 0.012 & 0.000 \end{bmatrix}$	0.941	0.270
B	$\begin{bmatrix} -5.240 & 0.016 \end{bmatrix}$	$\begin{bmatrix} 1.049 & 0.043 \end{bmatrix}$	$\begin{bmatrix} 1.535 & 0.078 \\ 0.078 & 0.013 \end{bmatrix}$	0.931	0.334
C	$\begin{bmatrix} 0.761 & 1.640 \end{bmatrix}$	$\begin{bmatrix} 1.434 & 0.256 \end{bmatrix}$	$\begin{bmatrix} 2.223 & 0.448 \\ 0.448 & 0.145 \end{bmatrix}$	0.624	0.510

Table 1: Three discretionary equilibria

possesses three discretionary equilibria. An implication of certainty equivalence[1] is that multiplicity of equilibrium is characterized by the deterministic component of the solution, $\{\pi_b, c_b, S_{bb}\}$. Usefully, equations (9)—(15) are recursive. We can solve equations (10), (13), and (15) for $\{c_b, \pi_b, S_{bb}\}$ and subsequently solve the remainder of the system for the stochastic component of the solution.

For an arbitrary private sector's response to the stock of debt, π_b, the policymaker's respoinse, described by c_b and S_{bb}, is given by equations (10) and (13). We find c_b and then the private sector's optimal response π_b^* is given by equation (15). Accordingly, for every possible π_b we can compute a unique π_b^* and plot the dependence $\pi_b^*(\pi_b)$. This relationship is shown in Figure 1, Panel I. Clearly, if $\pi_b = \pi_b^*$, then we have a fix-point and a discretionary equilibrium. For our baseline calibration the graph of $\pi_b^*(\pi_b)$ intersects the 45^o degree line at three points labelled A, B, and C. These three points represent three distinct discretionary equilibria.[2]

In Table 1, we report the policy rule, $c = \{c_\eta, c_b\}$, and the private-sector decision rules, $\pi = \{\pi_\eta, \pi_b\}$, for all three equilibria.

The three equilibria reported in Table 1 produce qualitatively and quantitatively different economic dynamics, as can be seen in Figure 1, Panel II, which shows the responses of key variables to a unit markup shock. Focusing first on equilibria A and B, inflation rises following the markup shock and the policy response is to defer consumption (by raising the nominal interest rate sufficiently high, this is implicit in our model). The decline in consumption lowers output and tax revenues, which leads to a rise in government debt. In subsequent periods, although

[1] See Currie and Levine (1993) or Backus and Driffill (1986) who show certainly equivalence for this class of models.

[2] The benchmark calibration follows Blake and Kirsanova (2012). The model's frequency is quarterly. The subjective discount factor β is set to 0.99, ρ is set to 0.9343, η is set to 0.1894, λ is set to 0.0582, ν is set to 0.0025, and ρ_u is set to 0.5.

interest rates are lowered to stimulate the economy and to bring it out of recession, government debt is brought back to baseline predominantly through fiscal surpluses, rather than through a decline in the cost of financing government debt.

In equilibrium C, monetary policy responds to the markup shock by stimulating consumption and output, which raises real marginal costs and causes inflation to rise by more than it otherwise would. This monetary policy causes tax revenues to rise and leads to a decline in government debt. To stabilize government debt, future policymakers raise the cost of financing government debt, which causes consumption, output, and real marginal costs to decline and places downward pressure on inflation. Specifically, in the spirit of Leeper (1991), monetary policy can be thought of as being active in equilibria A and B and passive in equilibrium C. Table 1 reveals this trade-off between the response to government debt and the response to the markup shock. Specifically, the more active the policy the more aggressively interest rates are raised in response to the markup shock.

In Table 1 the equilibria can be ranked by their unconditional loss $\mathbb{E}[L]$, which is reported in the last column. According to this ranking, equilibrium A is Pareto-preferred to the other two equilibria. But should we expect equilibrium A to prevail over equilibria B or C?

2.2 Equilibrium coordination

In this section we discuss two coordination mechanisms that can potentially reduce the number of empirically relevant equilibria. These mechanisms are expectational stability (Evans, 1986) and self-enforceability (Bernheim et al., 1987; Bernheim and Whinston, 1987). We relegate the formal treatment of these mechanisms for the general class of linear-quadratic models to Appendix A. Despite clear differences between these two coordination mechanisms, we show that the outcomes that they generate are inter-related. In particular, an equilibrium is self-enforceable if and only if it is iterative-expectationally stable under the private sector learning.

2.2.1 Learning and iterative-expectational stability

In order to arrive at a rational expectations equilibrium agents residing in an economy that are learning revise how they form expectations based on how these expectations affect the actual economy. In other words, these agents seek to rationalize, or equate, a perceived low-of-motion of the economy with the actual law-of-motion of the economy. If this "natural revision rule" returns the system to an equilibrium, then that equilibrium is said to be "expectationally stable", see Evans (1986). The required revisions occur in meta-time and constitute a learning process.

Like Evans (1986), Evans and Guesnerie (2003) and Evans and Guesnerie (2005) we view learning as a mechanism through which agents might coordinate on a discretionary equilibrium. In addition to private sector learning, we also look at the case where both private agents and the policymaker are learning.[3] The notion of stability under learning that we consider is iterative expectational stability (IE-stability).

[3] We prove in Appendix A that any equilibrium is IE-stable when the policymaker alone is learning so we do not consider it here.

Learning by private agents Recall that a discretionary equilibrium is fully characterized by the set $\{\pi_u,\ \pi_b,\ c_u,\ c_b,\ S_{uu},\ S_{ub},\ S_{bb}\}$. We want to examine whether private agents can learn their equilibrium reaction $\{\pi_u, \pi_b\}$, given the policy and payoffs described by $\{c_u, c_b, S_{uu}, S_{ub}, S_{bb}\}$.

Suppose private agents anticipate that the policymaker will implement equation (5) every period and that they employ the following perceived law of motion

$$\pi_t = \bar{\pi}_u u_t + \bar{\pi}_b b_t. \tag{16}$$

This perceived law-of-motion will be consistent with a rational expectations equilibrium if it is supported by the evolution of the economy. The evolution of the economy, equations (1)—(3), implies

$$\begin{aligned} \beta\left(\bar{\pi}_u \mathbb{E}_t u_{t+1} + \bar{\pi}_b \mathbb{E}_t b_{t+1}\right) &= \pi_u u_t + \pi_b b_t - \lambda\left(c_u u_t + c_b b_t\right) - \nu b_t - u_t \\ &= \beta\left(\bar{\pi}_u \rho u_t + \bar{\pi}_b\left(\rho b_t - \eta\left(c_u u_t + c_b b_t\right)\right)\right), \end{aligned} \tag{17}$$

and equating coefficients yields

$$\pi_u = \beta\bar{\pi}_u\rho - \beta\bar{\pi}_b\eta c_u + \lambda c_u + 1 \tag{18}$$

$$\pi_b = \beta\bar{\pi}_b\left(\rho - \eta c_b\right) + \lambda c_b + \nu. \tag{19}$$

Equations (18) and (19) define the revision-mapping, $\mathbb{T}$, from the initial guess of the decision rule $\bar{\pi} = \{\bar{\pi}_u, \bar{\pi}_b\}$ to the updated decision rule $\pi = \{\pi_u, \pi_b\}$, and can be summarized in the form $\pi = \mathbb{T}(\bar{\pi})$. A fixed point of this mapping results in a perceived law of motion for the economy that is consistent with the economy's actual law-of-motion in a discretionary equilibrium.

A fix-point, $\pi^* = \{\pi_u^*, \pi_b^*\}$ of the $\mathbb{T}$-map, $\pi = \mathbb{T}(\bar{\pi})$ is said to be locally IE-stable under private sector learning if

$$\lim_{k\to\infty} \mathbb{T}^k(\bar{\pi}) = \pi^*$$

for all $\bar{\pi}$ in a neighborhood of π^*, $\bar{\pi} \neq \pi^*$.

It follows that π^* is locally IE-stable if and only if it is a stable fix-point of the difference equation

$$\pi_{k+1} = \mathbb{T}(\pi_k) \tag{20}$$

where the index k denotes the step of the updating process.

In this model all discretionary equilibria are locally IE-stable under private sector learning. To see this, we linearize equation (20) around π^* to yield

$$\begin{bmatrix} \pi_u \\ \pi_b \end{bmatrix}_{k+1} = \begin{bmatrix} \beta\rho & -\beta\eta c_u \\ 0 & \beta\left(\rho - \eta c_b\right) \end{bmatrix} \begin{bmatrix} \pi_u \\ \pi_b \end{bmatrix}_k$$

Now applying standard results for linear difference equations, π^* is locally stable if and only if all of the eigenvalues of the derivative map $d\mathbb{T}(\pi^*) = \begin{bmatrix} \beta\rho & -\beta\eta c_u \\ 0 & \beta\left(\rho - \eta c_b\right) \end{bmatrix}$ have modulus less than one. The map $d\mathbb{T}(\pi^*)$ has two eigenvalues $z_1 = \beta\rho < 1$, and $z_2 = \beta\left(\rho - \eta c_b\right)$. To see that z_2 must also have modulus less than one, note that if a discretionary equilibrium exists then the rate at which debt is increasing over time, $\rho - \eta c_b$, cannot exceed $\beta^{-1/2}$. Therefore, existence of a discretionary equilibrium implies $|z_2| < 1$.

Joint learning Where the policy rule was taken as given in the analysis above here we adopt the alternative assumption that the policymaker is also learning. Thus, we assume that all agents seek to learn the discretionary equilibrium described by $\{\pi_u\pi_b, c_u, c_b, S_{uu}, S_{ub}, S_{bb}\}$. The monetary policymaker and the private sector form expectations about the discretionary equilibrium, and the perceived reaction of the private sector must be supported by the evolution of the economy in order to be consistent with a discretionary equilibrium.

The evolution of the economy, equations (1)—(3), implies

$$\pi_t = (\beta\rho\bar{\pi}_u + 1)\, u_t + (\beta\rho\bar{\pi}_b + \nu)\, b_t + (\lambda - \beta\eta\bar{\pi}_b)\, c_t \tag{21}$$

The perceived reaction of the policymaker must also be consistent with implementing the best response to the private sector's reaction function

$$\begin{aligned} & S_{uu}u_t^2 + 2S_{ub}u_t b_t + S_{bb}b_t^2 \\ = \; & \min_{c_t} \Big(\left((\beta\bar{\pi}_u\rho + 1)\, u_t + (\beta\bar{\pi}_b\rho + \nu)\, b_t + (\lambda - \beta\bar{\pi}_b\eta)\, c_t \right)^2 \\ & + \alpha c_t^2 + \beta \left(\bar{S}_{uu}\rho^2 u_t^2 + 2\bar{S}_{ub}\rho u_t (\rho b_t - \eta c_t) + \bar{S}_{bb} (\rho b_t - \eta c_t)^2 \right) \Big). \end{aligned} \tag{22}$$

The revised policy rule

$$c_t = c_u u_t + c_b b_t$$

with coefficients

$$c_u = -\frac{(\lambda - \beta\eta\bar{\pi}_b)(\beta\rho\bar{\pi}_u + 1) - \eta\rho\beta\bar{S}_{ub}}{\beta\eta^2\bar{S}_{bb} + (\lambda - \beta\eta\bar{\pi}_b)^2 + \alpha} = c_u\left(\bar{S}, \bar{\pi}\right) \tag{23}$$

$$c_b = -\frac{(\lambda - \beta\eta\bar{\pi}_b)(\beta\rho\bar{\pi}_b + \nu) - \eta\beta\rho\bar{S}_{bb}}{\beta\eta^2\bar{S}_{bb} + (\lambda - \beta\eta\bar{\pi}_b)^2 + \alpha} = c_b\left(\bar{S}, \bar{\pi}\right) \tag{24}$$

implements the best policy response. The value function is revised according to

$$S_{uu} = \pi_u\left(\bar{S}, \bar{\pi}\right)^2 + \alpha c_u\left(\bar{S}, \bar{\pi}\right)^2 + \beta\left(\rho^2\bar{S}_{uu} - 2\rho\eta\bar{S}_{ub}c_u\left(\bar{S}, \bar{\pi}\right) + \eta^2\bar{S}_{bb}c_u\left(\bar{S}, \bar{\pi}\right)^2\right) \tag{25}$$

$$\begin{aligned} S_{ub} = \; & \pi_u\left(\bar{S}, \bar{\pi}\right)\pi_b\left(\bar{S}, \bar{\pi}\right) + \alpha c_u\left(\bar{S}, \bar{\pi}\right)c_b\left(\bar{S}, \bar{\pi}\right) \\ & + \beta\rho\bar{S}_{ub}\left(\rho - \eta c_b\left(\bar{S}, \bar{\pi}\right)\right) - \beta\eta\bar{S}_{bb}c_u\left(\bar{S}, \bar{\pi}\right)\left(\rho - \eta c_b\left(\bar{S}, \bar{\pi}\right)\right) \end{aligned} \tag{26}$$

$$S_{bb} = \pi_b\left(\bar{S}, \bar{\pi}\right)^2 + \beta\bar{S}_{bb}\left(\rho - \eta c_b\left(\bar{S}, \bar{\pi}\right)\right)^2 + \alpha c_b\left(\bar{S}, \bar{\pi}\right)^2, \tag{27}$$

and the revision process for the private sector, described by (21), can be written as

$$\pi_t = \pi_u u_t + \pi_b b_t$$

where

$$\pi_u = \beta\bar{\pi}_u\rho + 1 + (\lambda - \beta\bar{\pi}_b\eta)\, c_u\left(\bar{S}, \bar{\pi}\right) = \pi_u\left(\bar{S}, \bar{\pi}\right), \tag{28}$$

$$\pi_b = \beta\bar{\pi}_b\rho + \nu + (\lambda - \beta\bar{\pi}_b\eta)\, c_b\left(\bar{S}, \bar{\pi}\right) = \pi_b\left(\bar{S}, \bar{\pi}\right), \tag{29}$$

and $c_u\left(\bar{S}, \bar{\pi}\right)$ and $c_b\left(\bar{S}, \bar{\pi}\right)$ are determined by equations (23)—(24).

Equations (23)—(29) define the $\mathbb{T}$-mapping, $x = \mathbb{T}(\bar{x})$, where $x = \{\pi_u, \pi_b, c_u, c_b, S_{uu}, S_{ub}, S_{bb}\}$. By construction any fixed point of this mapping is consistent with a discretionary equilibrium.

A fix-point, x^*, of the $\mathbb{T}$-map, $x = \mathbb{T}(\bar{x})$ is said to be locally IE-stable under joint learning if

$$\lim_{k\to\infty} \mathbb{T}^k(\bar{x}) = x^*$$

for all $\bar{x}$ in a neighborhood of x^*, $\bar{x} \neq x^*$. As before, the fixed point of the mapping needs to be locally Lyapunov-stable in order for all agents to learn *jointly* the discretionary equilibrium.

Substituting equations (23)—(24) into equations (28)—(27), allows the five eigenvalues of the derivative $\mathbb{T}$-map to be written as

$$\begin{aligned}
z_0 &= \beta\rho^2 < 1 \\
z_{1,2} &= \frac{1}{2}\beta\rho\frac{\zeta \pm \sqrt{\zeta^2 - 4\alpha\rho\left(\alpha + \beta\eta^2 S_{bb} + (\lambda - \beta\eta\pi_b)^2\right)}}{\alpha + S_{bb}\beta\eta^2 + (\lambda - \beta\eta\pi_b)^2} \\
z_{3,4} &= \frac{\delta^2 + \varsigma \pm \sqrt{\left(\delta^2 + \varsigma\right)^2 - 4\alpha\rho\delta^2\left(\alpha + \beta\eta^2 S_{bb} + (\lambda - \beta\eta\pi_b)^2\right)}}{2\left(\delta^2 + \varsigma\right)}
\end{aligned}$$

where

$$\begin{aligned}
\zeta &= \left(\alpha(1+\rho) + \beta\eta^2 S_{bb} + (\lambda\rho + \nu\eta)(\lambda - \beta\eta\pi_b)\right) \\
\delta &= \alpha\rho + \lambda^2\rho + \lambda\nu\eta - \beta\nu\eta^2\pi_b - \beta\lambda\eta\rho\pi_b \\
\varsigma &= \left(\alpha\left((\lambda - \beta\eta\pi_b)(\lambda\rho + 2\nu\eta + \beta\eta\rho\pi_b) + \alpha\rho + S_{bb}\beta\eta^2\rho\right) + 2S_{bb}\beta\eta^2(\lambda - \beta\eta\pi_b)(\lambda\rho + \nu\eta)\right)
\end{aligned}$$

A numerical examination of these eigenvalues establishes that only equilibria A and C are jointly learnable.

We note that the IE-stability properties associated with private sector learning and joint learning, although connected, are distinct. Joint learnability of an equilibrium neither implies nor is implied by private sector learnability of that equilibrium. In this model the joint learning criterion is more restrictive than the private sector learning criterion, but neither criterion discriminates between equilibria A and C.

2.2.2 Self-enforceability

We now approach the coordination problem by asking whether an equilibrium is self-enforceable (Bernheim et al., 1987; Bernheim and Whinston, 1987), robust to the potential formation of coalitions by sequential policymakers. Intuitively, policymakers can more easily coordinate on an equilibrium if that equilibrium is self-enforceable, and no group of policymakers finds it beneficial to form a coalition and deviate from equilibrium play. In the context of the debt model, there are three discretionary equilibria. These equilibria can also be viewed as symmetric Markov-perfect Nash equilibria of a Stackelberg game played between successive policymakers. Importantly, because the economic environment is one in which there is complete and perfect information, the existence and nature of all three equilibria is known to all agents. Moreover, the three equilibria can be welfare ranked and agents are not indifferent as to which equilibrium prevails.

We treat the policy rules associated with the three equilibria as a set of policy actions. Because the equilibria are Nash, if policymakers in periods $s = t+1, ..., \infty$ are expected to play $\{c_u^j, c_b^j\}$, $j = 1, 2, 3$, then the period-t policymaker's best response is to also play $\{c_u^j, c_b^j\}$. However, although it is never beneficial for the period-t policymaker to unilaterally deviate from Nash play, the period-t policymaker can potentially benefit from deviations that involve multiple policymakers. With this in mind, we introduce the possibility that a (finite) number of (sequential) policymakers could form a coalition that may collectively find it beneficial to deviate from the play prescribed in equilibrium j. The coalitions that we envisage are motivated by the fact that policymakers have tenures spanning multiple decision periods.[4]

For the sake of concreteness, consider the two equilibria, A and C. Both of these equilibria are not ruled out by the learning criteria that we considered above. Now, suppose the economy is in equilibrium C in period t. The loss associated with being in equilibrium C permanently is described by the value function matrix, S^C, whose elements are governed by equations (25)—(27). Suppose the period-t policymaker implements policy $\{c_u^A, c_b^A\}$, but that all agents anticipate that the policy $\{c_u^C, c_b^C\}$ will be played from period $t+1$ onwards. The private sector will react according to

$$\pi_u^{(t)} = \beta\rho\pi_u^{(t+1)} - \beta\eta\pi_b^{(t+1)}c_u^A + \lambda c_u^A + 1 \tag{30}$$

$$\pi_b^{(t)} = \beta\left(\rho - \eta c_b^A\right)\pi_b^{(t+1)} + \lambda c_b^A + \nu \tag{31}$$

where $\pi_b^{(t+1)} = \pi_b^C, \pi_u^{(t+1)} = \pi_u^C$, so the state in period $t+1$ will be given by

$$b_{t+1} = \left(\rho - \eta c_b^A\right) b_t - \eta c_u^A u_t. \tag{32}$$

Now, if the loss in period t is given by the value function, $S^{(t)}$, then it follows that

$$\begin{aligned} \begin{bmatrix} S_{uu}^{(t)} & S_{ub}^{(t)} \\ S_{ub}^{(t)} & S_{bb}^{(t)} \end{bmatrix} &= \begin{bmatrix} \pi_u^{(t)} & c_u^A \\ \pi_b^{(t)} & c_b^A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} \pi_u^{(t)} & \pi_b^{(t)} \\ c_u^A & c_b^A \end{bmatrix} \\ &\quad + \beta \begin{bmatrix} \rho_u & -\eta c_u^A \\ 0 & \left(\rho - \eta c_b^A\right) \end{bmatrix} \begin{bmatrix} S_{uu}^{(t+1)} & S_{ub}^{(t+1)} \\ S_{ub}^{(t+1)} & S_{bb}^{(t+1)} \end{bmatrix} \begin{bmatrix} \rho_u & 0 \\ -\eta c_u^A & \left(\rho - \eta c_b^A\right) \end{bmatrix} \end{aligned} \tag{33}$$

and the unconditional loss received by the period-t policymaker is

$$L^{CA} = trace\left(\begin{bmatrix} S_{uu}^{(t)} & S_{ub}^{(t)} \\ S_{ub}^{(t)} & S_{bb}^{(t)} \end{bmatrix} \begin{bmatrix} W_{uu} & W_{ub} \\ W_{ub} & W_{bb} \end{bmatrix} + \frac{\beta}{1-\beta} \begin{bmatrix} S_{uu}^{(t+1)} & S_{ub}^{(t+1)} \\ S_{ub}^{(t+1)} & S_{bb}^{(t+1)} \end{bmatrix} \begin{bmatrix} \sigma_u^2 & 0 \\ 0 & 0 \end{bmatrix} \right) \tag{34}$$

If L^{CA} is less than L^C, where L^C is the unconditional loss associated with equilibrium C, then the period-t policymaker would switch to playing the policy action associated with equilibrium A, and as all policymaker's face the same decision problem, this would imply that all policymakers would find it beneficial to switch to playing $\{c_u^A, c_b^A\}$.

[4] One might view the group of deviating policymakers to be small if it numbers less than a policymaker's average tenure. In the U. S., Federal Reserve chairmen are appointed to a four year term, but the average tenure is somewhat longer. In the U. K., monetary policy committee members have three-year contracts that overlap to prevent members from retiring simultaneously.

Characteristic	Equilibrium		
	A	B	C
(1) Average loss	0.2695	0.3344	0.5259
(2) IE-stable (Private sector)	yes	yes	yes
(3) IE-stable (Joint)	yes	no	yes
(4) Switch to eqm. A	—	35	18
(5) Self-enforceable	yes	no	no

Table 2: Equilibrium characteristics

Of course, by construction, such a switch can only happen if a coalition of policymakers jointly finds it beneficial to switch. Focusing on coalitions containing sequential policymakers, the unconditional loss received by the period-t policymaker can be computed using equations (30)—(34) as a recursion. Conceptually, if we have $L^{CA} < L^{C}$ for any finite coalition of sequential policymakers from period t onward, then the economy switches to equilibrium A, as equilibrium C is not self-enforceable.

Table 2 reports the characteristics of all three equilibria. Row (4) reports the minimum number of members in the coalition that will decide to leave equilibria B or C and switch to the Pareto-preferred equilibrium A. In particular, we find that if the economy is in equilibrium C, then a coalition containing just 18 successive policymakers will find it beneficial to switch to equilibrium A. Because all policymakers reason the same way, if 18 policymakers can enter into such a coalition, then all policymakers can enter into such coalitions and the economy will switch to equilibrium A.[5] It is more difficult to switch from equilibrium B, as a coalition containing 35 members is required, but equilibrium B is not jointly learnable. Equilibrium A is self-enforceable; this is reported in Row (5) in Table 2.

2.2.3 A connection between learnability and self-enforceability

IE-stability and self-enforceability are related. Using the general linear-quadratic rational expectations framework we prove in Appendix A that a finite coalition exists if and only if the Pareto-preferred equilibrium is locally IE-stable under private sector learning. The intuition for this result is that in order to generate a successful switch to the Pareto-preferred equilibrium the coalition of policymakers has to induce a switch in private-sector expectations about the future policy.

3 A DSGE model

Following Woodford (2003, Ch.5), the economy is populated by households, intermediate-good producing firms, final-good producing firms, and a central bank. Households are identical and infinitely lived, choosing consumption, c_t, labor, l_t, and nominal holdings of next period bonds,

[5] With the model parameterized to a quarterly frequency, 18 periods represents four and half years, which is in the realm of the typical tenure for a policymaker. This suggests that the members of the coalition could consist of the policymakers associated with the tenure of a single central banker.

b_{t+1}, to maximize expected discounted utility subject to a budget constraint. On the production side, a unit-continuum of monopolistically competitive intermediate-good producing firms, indexed by $\omega \in [0,1]$, produce by combining labor services hired in a perfectly competitive market with their firm-specific capital. These intermediate-good producing firms make labor and investment decisions, seeking to maximize their value subject to their production technology

$$Y_t(\omega) = e^{u_t} K_t(\omega)^{\alpha} L_t(\omega)^{(1-\alpha)},$$

their capital accumulation equation

$$I_t(\omega) = I\left(\frac{K_{t+1}(\omega)}{K_t(\omega)}\right) K_t(\omega),$$

where $I(1) = \delta$, $I'(1) = 1$, and $I''(1) = \eta$, and a Calvo (1983) price rigidity, where firms that cannot optimally set their price in a given period are assumed to index their price to lagged aggregate inflation (Smets and Wouters, 2003). Profits are aggregated and returned to households (shareholders) in the form of a lump-sum dividend. The final-good producing firms purchase intermediate goods, aggregate them into a final good according to a Dixit and Stiglitz (1977) production technology, and sell these final goods in a perfectly competitive market to households and firms to consume and invest, respectively.

After aggregating and log-linearizing about a zero-inflation nonstochastic steady state, the model's constraints and first-order conditions are

$$\begin{aligned}
\pi_t &= \frac{\beta}{1+\theta\beta}\mathbb{E}_t\pi_{t+1} + \frac{\theta}{1+\theta\beta}\pi_{t-1} + \frac{(1-\xi)(1-\beta\xi)}{(1+\theta\beta)\xi}mc_t + u_t, \\
c_t &= \mathbb{E}_t c_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\pi_{t+1} - g_t + \mathbb{E}_t g_{t+1}\right), \\
k_{t+1} &= \frac{1}{1+\beta}k_t + \frac{\beta}{1+\beta}\mathbb{E}_t k_{t+2} + \frac{1-\beta(1-\delta)}{(1+\beta)\eta}\mathbb{E}_t ms_{t+1} - \frac{1}{(1+\beta)\eta}\left(i_t - \mathbb{E}_t\pi_{t+1}\right) \\
mc_t &= w_t - y_t + l_t, \\
w_t &= \chi l_t + \sigma c_t - g_t, \\
y_t &= (1-\gamma)c_t + \frac{\gamma}{\delta}\left(k_{t+1} - (1-\delta)k_t\right), \\
y_t &= v_t + \alpha k_t + (1-\alpha)l_t, \\
ms_t &= w_t - k_t + l_t
\end{aligned}$$

where $\beta \in (0,1)$ is the discount factor, $\rho \equiv \frac{1-\beta}{\beta}$ is the discount rate, $\gamma \equiv \frac{\alpha\delta}{\rho+\delta}\frac{\varepsilon-1}{\varepsilon}$ is the steady-state share of investment in output, $\varepsilon > 1$ is the steady-state elasticity of substitution between intermediate goods, $\delta \in (0,1)$ is the depreciation rate, and $\eta > 0$ is the elasticity of the investment-to-capital ratio with respect to Tobin's q evaluated at steady state (Eichenbaum and Fisher, 2007).

Although the model allows for three stochastic elements: an aggregate consumption-preference shock, g_t; an aggregate markup shock, u_t; and an aggregate technology shock, v_t, we zero-out g_t and v_t in order to focus on the policy trade-offs associated with the markup shock, u_t.[6]

[6] To parameterize the model, we set the discount factor, β, to 0.99, the Calvo price rigidity, ξ, to 0.75, the inflation indexation parameter, θ, to 0.60, the Cobb-Douglas production function parameter, α, to 0.36, the capital adjustment costs parameter to 6.0, the labor supply elasticity, χ, to 1, the elasticity of intertemporal substitution, σ, to 2, the depreciation rate, δ, to 0.025, the elasticity of substitution between goods, ε, to 11, and the shock persistence, ρ_u, to 0.3.

The central bank's loss function is assumed to have the form

$$L_t = \mathbb{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\pi_s^2 + \frac{(1-\xi)(1-\beta\xi)}{(1+\theta\beta)\xi\varepsilon} y_s^2 \right).$$

This loss is, in a certain sense, *ad hoc*, but the precise form of the objective function is not essential for our results.

Monetary policy aims to stabilize inflation and does this via influencing the path for marginal cost. Adapting a result from Dennis and Sóderstróm (1990), the forward representation of the inflation equation is given by

$$\pi_t = \theta\pi_{t-1} + \frac{(1-\xi)(1-\beta\xi)}{\xi} \mathbb{E}_t \sum_{s=t}^{\infty} \beta^{s-t} mc_s + \frac{1+\theta\beta}{1-\rho_u\beta} u_t. \tag{35}$$

Moreover, real marginal costs can be expressed as

$$mc_t = \left(\frac{\alpha+\chi}{1-\alpha} + \frac{\sigma}{1-\gamma} \right) y_t + \left(\frac{\sigma\gamma(1-\delta)}{(1-\gamma)\delta} - \frac{\alpha(\alpha+\chi)}{1-\alpha} \right) k_t - \frac{\sigma\gamma}{(1-\gamma)\delta} k_{t+1}. \tag{36}$$

It is apparent that movements in mc_t and mc_{t+1} are highly substitutable in terms of their effect on π_t and that, for any initial value of inflation, there are multiple paths for mc_t that will return inflation to target. These different paths for real marginal costs are associated with different monetary policies and with different performance in terms of loss. Equation (36) shows that monetary policy can affect mc_t through two distinct channels. To lower real marginal costs, the central bank can raise the real interest rate, weakening aggregate demand and thereby causing y_t to decline or it can lower the real interest rate to stimulate investment and thereby boost the future capital stock. Notice that raising (lowering) the real interest rate causes both y_t and k_{t+1} to decline (rise) and that y_t and k_{t+1} have countervailing effects on mc_t. As a consequence, the desirability of each policy from the perspective of the period-t policymaker turns on how future policymakers are expected to respond to movements in the capital stock.

Consider the case where future policymakers are expected to lower the interest rate in response to a rise in the capital stock. Following a positive markup shock, the policy of raising the real interest rate and causing y_t and k_{t+1} to decline will successfully deliver lower real marginal costs and inflation because the boost in future real marginal costs caused by the decline in the capital stock is offset by higher interest rates in the future. Under this approach, monetary policy responds to the positive markup shock by contracting demand, lowering real marginal costs and inflation, and by then lowering interest rates as inflation declines allowing the economy to recover, producing an equilibrium. Alternatively, if future policymakers are expected to raise the interest rate in response to a higher capital stock, then a policy that lowers the real interest rate and stimulates investment can bring about a decline in inflation, despite the boost to y_t and mc_t today, because future policymakers respond to the higher capital stock by tightening monetary policy, producing another equilibrium.

More formal investigation demonstrates that there are three discretionary equilibria, an equilibrium in which the real interest rate is raised in response to the markup shock which we label as equilibrium A, an equilibrium in which the real interest rate is lowered which we label as equilibrium C, and a 'middle' equilibrium in which the interest rate is only weakly raised which we

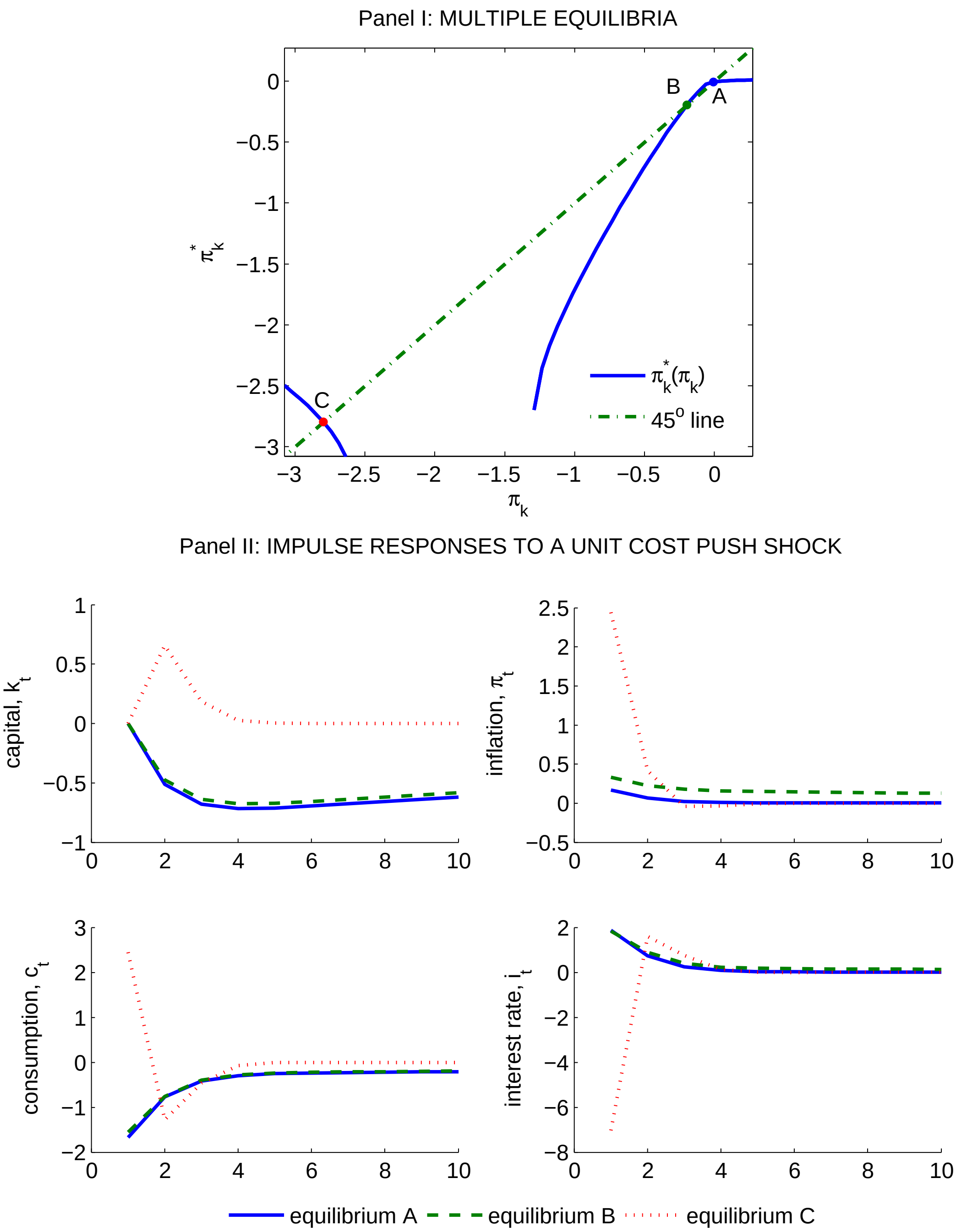

Figure 2: A DSGE model with inflation inertia

Characteristic	Equilibrium		
	A	B	C
(1) Average loss	0.2436	0.7653	6.5921
(2) IE-stable (Private sector)	yes	yes	yes
(3) IE-stable (Joint)	yes	no	yes
(4) Switch to eqm. A	—	3	69
(5) Self-enforceable	yes	no	no

Table 3: Equilibrium characteristics

label as equilibrium B. We demonstrate their existence by looking for a fix-point in the private sector's response to the capital stock and plot them in Figure 2, Panel I.

The economy's behavior in the different equilibria are shown in 2, Panel II, which displays the responses of key variables to a unit markup shock. Focusing first on equilibria A and B, following the markup shock the interest rate is raised by more than the increase in inflation, causing the real interest rate to rise. The higher real interest rate generates a decline in consumption and investment, which lowers output and real marginal costs. Further, the fall in investment leads to a decline in the capital stock. In subsequent periods, the decline in real marginal costs causes inflation to moderate. With inflation declining back to baseline, monetary policy responds by lowering the interest rate and stimulating demand. In these two equilibria, monetary policy stabilizes the economy in the traditional way, contracting output and hence real marginal costs in order to keep inflationary pressures contained.

In contrast, in equilibrium C the interest rate is lowered in response to the positive markup shock, generating a big decline in the real interest rate. The lower real interest rate stimulates consumption and investment, which pushes up output and real marginal costs and further boosts inflation. However, the rise in investment causes the capital stock to increase and the capital build up eventually lowers real marginal costs while inducing tighter monetary policy. Although the policy tightening is aimed primarily at lowering investment, it also serves to lower output, which causes a further decline in real marginal costs. In this equilibrium, monetary policy responds to the markup shock by stimulating the economy in order to boost capital spending. This policy succeeds in stabilizing the economy because the higher capital stock causes future real marginal costs to decline and future monetary policy to tighten.

As in the previous example, the economy behaves very differently in equilibria A and B than it does in equilibrium C. The conventional policy associated with equilibrium A is welfare superior to the unconventional policy associated with equilibrium C, but the model does not suggest which equilibrium is likely to realize. We apply the same coordination mechanisms as in the previous section to understand if and how the Pareto-preferred equilibrium can come about.

The results are presented in Table 3. The conventional policy is superior to the unconventional policy, see Row (1). Rows (2) and (3) show that both equilibrium A and equilibrium C are jointly learnable and learnable by private agents. However, because the Pareto-preferred equilibrium (equilibrium A) is private-sector learnable, it follows that equilibria B and C are not self-enforceable. Row (4) in Table 3 reports the minimum number of members in the coalition which will decide to leave equilibria B and C and switch to the Pareto-optimal equilibrium A.

The switch to the Pareto-preferred equilibrium presents much bigger challenge for the policymaker in this model than in the previous one as the size of the minimal coalition is large and the tenure in the office required to switch from equilibrium C is close to 20 years. As previously, the Pareto-preferred equilibrium is self-enforceable, as reported in Row (5).

4 Conclusion

Discretionary policymakers can manage neither the expectations of private agents nor the actions of future policymakers. As a consequence, discretionary policymakers are susceptible to expectations traps and coordination failures and discretionary control problems can have multiple equilibria. Recognizing this potential for multiple equilibria, this paper addresses the important issue of equilibrium coordination. The paper's main contribution is to develop several equilibrium coordination mechanisms, mechanisms motivated by expectational stability and self-enforceability. These mechanisms do not require any ability of the policymaker to precommit to any sort of actions.

Although these coordination mechanisms happen to point to the Pareto-preferred equilibrium as the equilibrium of interest, this need not have been the case. Our experience is that the Pareto-preferred equilibrium is jointly learnable, but that it is not necessarily private sector learnable. It is entirely possible, therefore, that in other models these coordination mechanisms could point toward equilibria (or an equilibrium) that is Pareto-dominated.

Finally, while we have described and applied several coordination mechanisms in this paper, there are, of course, other approaches to determining among equilibria. One such approach might be to determine an equilibrium of interest using minimax-loss or minimax-regret; another might be to identify an equilibrium from the limiting behavior of quasi-commitment policies. We leave the study and application of these criteria, and an investigation into whether multiple discretionary equilibria is a general feature of New Keynesian monetary policy models, to future work.

A Coordination mechanisms in LQ RE Models

A.1 The discretionary control problem

In this appendix, we outline the control problem facing a discretionary policymaker in the general linear-quadratic rational expectations framework. We then reinterpret this control problem as a non-cooperative dynamic game and show that the standard optimal discretionary policy is a symmetric Markov-perfect Nash equilibrium of a dynamic game in which the policymaker is a Stackelberg leader and private agents are followers. To make explicit the game's leadership structure, we call this equilibrium a symmetric Markov-perfect Stackelberg-Nash equilibrium. Finally, we show that solving for a symmetric Markov-perfect Stackelberg-Nash equilibrium in this game requires solving a particular fix-point problem.

A.1.1 Constraints and objectives

The economic environment is one in which n_1 predetermined variables, $\mathbf{x}_t$, and n_2 nonpredetermined variables, $\mathbf{y}_t$, $t = 0, 1, ..., \infty$, evolve over time according to

$$\begin{aligned}
\mathbf{x}_{t+1} &= \mathbf{A}_{11}\mathbf{x}_t + \mathbf{A}_{12}\mathbf{y}_t + \mathbf{B}_1\mathbf{u}_t + \mathbf{v}_{\mathbf{x}t+1}, &(37)\\
\mathbb{E}_t\mathbf{y}_{t+1} &= \mathbf{A}_{21}\mathbf{x}_t + \mathbf{A}_{22}\mathbf{y}_t + \mathbf{B}_2\mathbf{u}_t, &(38)
\end{aligned}$$

where $\mathbf{u}_t$ is a $p \times 1$ vector of control variables, $\mathbf{v}_{\mathbf{x}t} \sim i.i.d.\,[\mathbf{0}, \mathbf{\Sigma}]$ is an $v \times 1$ $(1 \leq v \leq n_1)$ vector of white-noise innovations, and $\mathbb{E}_t$ is the mathematical expectations operator conditional upon period t information. Equations (37) and (38) capture aggregate constraints and technologies and the behavior (aggregate first-order conditions) of private agents. For their part, private agents are comprised of households and firms who are ex ante identical, respectively, infinitely lived, and atomistic. The matrices $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$, $\mathbf{A}_{22}$, $\mathbf{B}_1$, and $\mathbf{B}_2$ are conformable with $\mathbf{x}_t$, $\mathbf{y}_t$, and $\mathbf{u}_t$ as necessary and contain the parameters that govern preferences and technologies. Importantly, the matrix $\mathbf{A}_{22}$ is assumed to have full rank.

In addition to private agents, the economy is populated by a large player, a policymaker. For each period t, the period-t policymaker's objectives are described by the loss function

$$L_t = \mathbb{E}_t \sum_{k=t}^{\infty} \beta^{(k-t)} \left[\mathbf{z}_k^{'}\mathbf{W}\mathbf{z}_k + 2\mathbf{z}_k^{'}\mathbf{U}\mathbf{u}_k + \mathbf{u}_k^{'}\mathbf{Q}\mathbf{u}_k\right], \qquad (39)$$

where $\beta \in (0, 1)$ is the discount factor and $\mathbf{z}_k = \begin{bmatrix} \mathbf{x}_k^{'} & \mathbf{y}_k^{'} \end{bmatrix}^{'}$. We assume that the weighting matrices $\mathbf{W}$ and $\mathbf{Q}$ are symmetric and, to ensure that the loss function is convex, that the matrix $\begin{bmatrix} \mathbf{W} & \mathbf{U} \\ \mathbf{U}^{'} & \mathbf{Q} \end{bmatrix}$ is positive semi-definite.[7] We assume that the policymaker is a Stackelberg leader and that private agents are followers; we further assume that the policymaker does not have access to a commitment technology and that policy is conducted under discretion.[8] With policy conducted under discretion, the policymaker sets its control variables, $\mathbf{u}_t$, each period to minimize equation (39), taking the state, $\mathbf{x}_t$, and the decision rules of all future agents as given. Since the policymaker is a Stackelberg leader, the period-t policy decision is formulated taking equation (38) as well as equation (37) into account.

The control problem described above has many of the characteristics of an infinite horizon non-cooperative dynamic game. Following Oudiz and Sachs (1985) and Cohen and Michel (1988), the strategic players in the game are the (infinite) sequence of policymakers with private agents behaving competitively. Although individual private agents are not strategic players in aggregate they are not inconsequential. Private agents are important because private-sector expectations are the conduit through which strategic interaction between current and future policymakers occurs. In this decision problem, policy behavior is described by a policy strategy, private-agent

[7] It is standard to assume that the weighting matrices, $\mathbf{W}$ and $\mathbf{Q}$, are symmetric positive semi-definite and symmetric positive definite, respectively (see Anderson et al. (1996), for example). However, since many economic applications involve a loss function that places no penalty on the control variables, we note that the requirement of $\mathbf{Q}$ being positive definite can be weakened to $\mathbf{Q}$ being positive semi-definite if additional assumptions about other system matrices are met (Clements and Wimmer, 2003).

[8] Events within a period occur as follows. After observing the state, $\mathbf{x}_t$, decisions are made first by the incumbent policymaker and subsequently by private agents. At the end of the period the shocks $\mathbf{v}_{\mathbf{x}t+1}$ are realized.

behavior is described by a private sector strategy, the expectations operator ($\mathbb{E}_t$) and policy loss (payoff) are induced by the policy and private sector strategies, and the equilibrium that we seek to analyze is a symmetric Markov-perfect Stackelberg-Nash equilibrium.

A.1.2 Some useful definitions and equilibrium concepts

In the previous section we emphasized that the discretionary control problem can be modeled as a non-cooperative dynamic game, with the decisions of the policymaker and of private agents taking the form of strategies. Further, we noted that because the policymaker is assumed to be an intra-period leader the discretionary equilibrium that we are interested in is a symmetric Markov-perfect Stackelberg-Nash equilibrium. We now make these terms precise.[9]

Definition 1 *A policy strategy* $\mathbf{S}$ *is a sequence of policy rules* $\{\mathbf{F}_t\}_0^\infty$*, where* $\mathbf{F}_t$ *is a function that maps* $\{\mathbf{x}_t\}_0^t$ *to* $\mathbf{u}_t$*. A policy strategy is said to be a Markov policy strategy if and only if each policy rule* $\mathbf{F}_t$ *is a function that maps* $\mathbf{x}_t$ *to* $\mathbf{u}_t$*. We denote by* $\mathbf{S}_{-t}$ *the sequence of policy rules* $\{\mathbf{F}_s\}_0^\infty$ *excluding* $\mathbf{F}_t$*.*

Definition 2 *A private sector strategy* $\mathbf{T}$ *is a sequence of decision rules* $\{\mathbf{H}_t\}_0^\infty$*, where* $\mathbf{H}_t$ *is a function that maps* $\{\mathbf{x}_t\}_0^t$ *to* $\mathbf{y}_t$*. A private sector strategy is said to be a Markov private sector strategy if and only if each decision rule* $\mathbf{H}_t$ *is a function that maps* $\mathbf{x}_t$ *to* $\mathbf{y}_t$*. We denote by* $\mathbf{T}_{-t}$ *the sequence of decision rules* $\{\mathbf{H}_s\}_0^\infty$ *excluding* $\mathbf{H}_t$*.*

Definition 3 *A policy strategy* $\mathbf{S}$ *is a Stackelberg-Nash equilibrium if for every decision period* t*: i)* $\mathbf{F}_t$ *minimizes equation (39) subject to equations (37) and (38) and* $\mathbf{x}_t$ *known, taking* $\mathbf{S}_{-t}$ *and* $\mathbf{T}_{-t}$ *as given; and ii)* $\mathbf{H}_t$ *satisfies equations (37) and (38), taking* $\mathbf{S}$ *and* $\mathbf{T}_{-t}$*, as given.*

Definition 4 *A policy strategy* $\mathbf{S}$ *is a perfect Stackelberg-Nash equilibrium if for every decision period* t *and any history* $\{\mathbf{F}_s, \mathbf{H}_s\}_0^{t-1}$*: i)* $\mathbf{F}_t$ *minimizes equation (39) subject to equations (37) and (38) and* $\mathbf{x}_t$ *known, taking* $\mathbf{S}_{-t}$ *and* $\mathbf{T}_{-t}$ *as given; and ii)* $\mathbf{H}_t$ *satisfies equations (37) and (38), taking* $\mathbf{S}$ *and* $\mathbf{T}_{-t}$ *as given.*

A perfect Stackelberg-Nash equilibrium is time-consistent because it is subgame perfect. However, the strategies that characterize equilibrium are not necessarily Markov strategies and, as a consequence, trigger-strategy equilibria, and other equilibria supported by threats and punishments are not ruled out. The sustainable equilibria studied by Chari and Kehoe (1990), Ireland (1997), and Kurozumi (2008) as well as the "reputational" equilibria examined by Barro and Gordon (1983) are all examples of perfect Stackelberg-Nash equilibria.

Definition 5 *A policy strategy* $\mathbf{S}$ *is a Markov-perfect Stackelberg-Nash equilibrium if restricting* $\mathbf{S}$ *to be a Markov policy strategy and* $\mathbf{T}$ *to be a Markov private sector strategy, for every time period*

[9] Although the discretionary control problem described in Section A.1 is standard in the monetary policy literature (it is the formulation used by Clarida et al. (1999), for example) there are other notions of discretion in the literature. These different notions of discretion are associated either with different dynamic games or with different equilibrium concepts. Cohen and Michel (1988), de Zeeuw and van der Ploeg (1991), and Chow (1997, Ch. 6), provide useful discussions.

t and any history of Markov policy and decision rules $\{\mathbf{F}_s, \mathbf{H}_s\}_0^{t-1}$: i) $\mathbf{F}_t$ minimizes equation (39) subject to equations (37) and (38) and $\mathbf{x}_t$ known, taking $\mathbf{S}_{-t}$ and $\mathbf{T}_{-t}$ as given; and ii) $\mathbf{H}_t$ satisfies equations (37) and (38), taking $\mathbf{S}$ and $\mathbf{T}_{-t}$ as given.

Definition 6 *A policy strategy $\mathbf{S}$ is a symmetric Markov-perfect Stackelberg-Nash equilibrium if and only if: i) $\mathbf{S}$ is a Markov-perfect Nash equilibrium in which $\mathbf{F}_t = \mathbf{F}$, $\forall\ t$; and ii) $\mathbf{T}$ is a Markov private sector strategy in which $\mathbf{H}_t = \mathbf{H}$, $\forall\ t$.*

A.1.3 Characterizing equilibrium

For the decision problem summarized by equations (37)—(39), we now describe the equilibrium conditions that characterize a symmetric Markov-perfect Stackelberg-Nash equilibrium, focusing on equilibria for which the decision rules are linear in the state vector.

First, if a symmetric Markov-perfect Stackelberg-Nash equilibrium exists, then in this equilibrium the behavior of the policymaker and private agents in all states, $\mathbf{x}_t$, and in all decision periods, $t = 0, ..., \infty$, is described by the linear rules

$$\mathbf{u}_t = \mathbf{F}\mathbf{x}_t, \tag{40}$$

$$\mathbf{y}_t = \mathbf{H}\mathbf{x}_t, \tag{41}$$

respectively. In this equilibrium, the law-of-motion for the predetermined variables is given by

$$\mathbf{x}_{t+1} = \mathbf{M}\mathbf{x}_t + \mathbf{v}_{\mathbf{x}t+1},$$

where the spectral radius of $\mathbf{M}$ is less than $\beta^{-\frac{1}{2}}$. Further, since the loss function is quadratic and the constraints are linear, the payoff to the policymaker in period t that corresponds to these rules is summarized by the quadratic state-contingent value function

$$V(\mathbf{x}_t) = \mathbf{x}_t^{'}\mathbf{V}\mathbf{x}_t + d,$$

where $\mathbf{V}$ is symmetric positive semi-definite. Importantly, because the policy rule, $\mathbf{F}$, and the decision rule, $\mathbf{H}$, in a symmetric Markov-perfect Stackelberg-Nash equilibrium apply in all states, the subgames one needs to consider when solving for a symmetric Markov-perfect Stackelberg-Nash equilibrium are those indexed only by time.

Second, if a symmetric Markov-perfect Stackelberg-Nash equilibrium exists for the subgame beginning in period $t+1$, then one can condition the subgame beginning in period t on the $\widetilde{\mathbf{H}}$, $\widetilde{\mathbf{F}}$, $\widetilde{\mathbf{M}}$, $\widetilde{\mathbf{V}}$, and $\tilde{d}$ that characterize the equilibrium of the subgame beginning in period $t+1$. Thus, the decision problem facing the policymaker in the subgame beginning in period t is to choose a rule for setting $\mathbf{u}_t$ in order to minimize

$$\begin{aligned}\mathbf{x}_t^{'}\mathbf{V}\mathbf{x}_t + d &= \mathbf{x}_t^{'}\mathbf{W}_{11}\mathbf{x}_t + \mathbf{y}_t^{'}\mathbf{W}_{22}\mathbf{y}_t + \mathbf{u}_t^{'}\mathbf{Q}\mathbf{u}_t + 2\mathbf{x}_t^{'}\mathbf{W}_{12}\mathbf{y}_t \\ &\quad +2\mathbf{x}_t^{'}\mathbf{U}_1\mathbf{u}_t + 2\mathbf{y}_t^{'}\mathbf{U}_2\mathbf{u}_t + \beta\mathbb{E}_t\left(\mathbf{x}_{t+1}^{'}\widetilde{\mathbf{V}}\mathbf{x}_{t+1} + \tilde{d}\right),\end{aligned} \tag{42}$$

subject to equations (37) and (38) and

$$\mathbf{u}_{t+1} = \widetilde{\mathbf{F}}\mathbf{x}_{t+1}, \tag{43}$$

$$\mathbf{y}_{t+1} = \widetilde{\mathbf{H}}\mathbf{x}_{t+1}, \tag{44}$$

and $\mathbf{x}_t$ known. Importantly, although $\widetilde{\mathbf{H}}$ and $\widetilde{\mathbf{V}}$ are functions of $\widetilde{\mathbf{F}}$, the problem's structure means that $\widetilde{\mathbf{F}}$ does not have a separate, explicit, effect on the current period payoff, $V(\mathbf{x}_t) = \mathbf{x}_t^{'}\mathbf{V}\mathbf{x}_t + d$. Consequently, as this decision problem is formulated, equation (43) does not bind as a separate constraint.

Using equation (44) to form $\mathbb{E}_t\mathbf{y}_{t+1}$, substituting the resulting expression into equation (38), and exploiting equation (37), we obtain the aggregate private-sector reaction function

$$\mathbf{y}_t = \mathbf{J}\mathbf{x}_t + \mathbf{K}\mathbf{u}_t, \tag{45}$$

where

$$\mathbf{J} = \left(\mathbf{A}_{22} - \widetilde{\mathbf{H}}\mathbf{A}_{12}\right)^{-1}\left(\widetilde{\mathbf{H}}\mathbf{A}_{11} - \mathbf{A}_{21}\right), \tag{46}$$

$$\mathbf{K} = \left(\mathbf{A}_{22} - \widetilde{\mathbf{H}}\mathbf{A}_{12}\right)^{-1}\left(\widetilde{\mathbf{H}}\mathbf{B}_{1} - \mathbf{B}_{2}\right). \tag{47}$$

Provided $rank\,(\mathbf{K}) \neq \mathbf{0}$, equation (45) implies that the period-t policymaker is a Stackelberg leader with respect to the period-t private sector. Then, substituting equation (45) into equations (42) and (37), the decision problem facing the policymaker in the subgame beginning in period t is to choose a rule for setting $\mathbf{u}_t$ in order to minimize

$$\mathbf{x}_t^{'}\mathbf{V}\mathbf{x}_t + d = \mathbf{x}_t^{'}\widehat{\mathbf{W}}\mathbf{x}_t + 2\mathbf{x}_t^{'}\widehat{\mathbf{U}}\mathbf{u}_t + \mathbf{u}_t^{'}\widehat{\mathbf{Q}}\mathbf{u}_t + \beta\mathbb{E}_t\left(\mathbf{x}_{t+1}^{'}\widetilde{\mathbf{V}}\mathbf{x}_{t+1} + \tilde{d}\right), \tag{48}$$

subject to

$$\mathbf{x}_{t+1} = \widehat{\mathbf{A}}\mathbf{x}_t + \widehat{\mathbf{B}}\mathbf{u}_t + \mathbf{v}_{\mathbf{x}t+1}, \tag{49}$$

where

$$\widehat{\mathbf{W}} = \mathbf{W}_{11} + \mathbf{W}_{12}\mathbf{J} + \mathbf{J}^{'}\mathbf{W}_{21} + \mathbf{J}^{'}\mathbf{W}_{22}\mathbf{J}, \tag{50}$$

$$\widehat{\mathbf{U}} = \mathbf{W}_{12}\mathbf{K} + \mathbf{J}^{'}\mathbf{W}_{22}\mathbf{K} + \mathbf{U}_{1} + \mathbf{J}^{'}\mathbf{U}_{2}, \tag{51}$$

$$\widehat{\mathbf{Q}} = \mathbf{Q} + \mathbf{K}^{'}\mathbf{W}_{22}\mathbf{K} + 2\mathbf{K}^{'}\mathbf{U}_{2}, \tag{52}$$

$$\widehat{\mathbf{A}} = \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{J}, \tag{53}$$

$$\widehat{\mathbf{B}} = \mathbf{B}_{1} + \mathbf{A}_{12}\mathbf{K}. \tag{54}$$

Conditional on $\widetilde{\mathbf{H}}$ and $\widetilde{\mathbf{V}}$ (and $\widetilde{\mathbf{F}}$), equations (48) and (49) describe a standard linear-quadratic dynamic programming problem. To guarantee existence of a solution, we need $\left(\widehat{\mathbf{A}}, \widehat{\mathbf{B}}\right)$ to be a controllable pair and $\left(\widehat{\mathbf{A}}, \widehat{\mathbf{W}}\right)$ to be a detectable pair (Laub (1979), Anderson et al. (1996)). Suppose that, for a given $\mathbf{J}$ and $\mathbf{K}$, $\left(\widehat{\mathbf{A}}, \widehat{\mathbf{B}}\right)$ is a controllable pair and $\left(\widehat{\mathbf{A}}, \widehat{\mathbf{W}}\right)$ is a detectable pair, then the solution to the subgame beginning in period t has the form of rules (40) and (41), with

$$\mathbf{F} = -\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}^{'}\widetilde{\mathbf{V}}\widehat{\mathbf{B}}\right)^{-1}\left(\widehat{\mathbf{U}}^{'} + \beta\widehat{\mathbf{B}}^{'}\widetilde{\mathbf{V}}\widehat{\mathbf{A}}\right), \tag{55}$$

$$\mathbf{0} = \widetilde{\mathbf{H}}\mathbf{A}_{12}\mathbf{H} - \mathbf{A}_{22}\mathbf{H} + \widetilde{\mathbf{H}}\left(\mathbf{A}_{11} + \mathbf{B}_{1}\mathbf{F}\right) - \mathbf{A}_{21} - \mathbf{B}_{2}\mathbf{F}, \tag{56}$$

$$\mathbf{V} = \widehat{\mathbf{W}} + 2\widehat{\mathbf{U}}\mathbf{F} + \mathbf{F}^{'}\widehat{\mathbf{Q}}\mathbf{F} + \beta\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)^{'}\widetilde{\mathbf{V}}\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right), \tag{57}$$

$$d = \beta tr\left(\mathbf{V}\boldsymbol{\Sigma}\right) + \beta\tilde{d}. \tag{58}$$

From $\mathbf{F}$ and $\mathbf{H}$, the matrix $\mathbf{M}$ in the law-of-motion for the predetermined variables is then given by

$$\mathbf{M} = \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H} + \mathbf{B}_1\mathbf{F}. \tag{59}$$

Because $\widetilde{\mathbf{H}}$, $\widetilde{\mathbf{F}}$, $\widetilde{\mathbf{M}}$, $\widetilde{\mathbf{V}}$, and $\tilde{d}$ represent a symmetric Markov-perfect Stackelberg-Nash equilibrium for the subgame beginning in period $t+1$, any fix-point of equations (55)—(59) in which $\mathbf{H} = \widetilde{\mathbf{H}}$, $\mathbf{F} = \widetilde{\mathbf{F}}$, $\mathbf{M} = \widetilde{\mathbf{M}}$, $\mathbf{V} = \widetilde{\mathbf{V}}$, and $d = \tilde{d}$, such that $\mathbf{V}$ is symmetric positive semi-definite and $\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right)$ has full rank, is a symmetric Markov-perfect Stackelberg-Nash equilibrium for the subgame beginning in period t.

A.2 Coordination mechanisms

A.2.1 Learning and expectational stability

Recall that a symmetric Markov-perfect Nash equilibrium is characterized by $\{\mathbf{H},\ \mathbf{F},\ \mathbf{M},\ \mathbf{V},\ d\}$. Because $\mathbf{M}$ and d follow immediately and uniquely from $\mathbf{F}$, $\mathbf{H}$, and $\mathbf{V}$, we implement the partitioning $\{\{\mathbf{H}, \mathbf{F}, \mathbf{V}\}, \{\mathbf{M}, d\}\}$ and focus on $\{\mathbf{H}, \mathbf{F}, \mathbf{V}\}$ in what follows. Specifically, we consider:

1. Private sector learning, where we analyze whether private agents can learn $\mathbf{H}$, conditional on $\{\mathbf{F}, \mathbf{V}\}$.
2. Policymaker learning, where we analyze whether the policymaker can learn $\{\mathbf{F}, \mathbf{V}\}$, conditional on $\{\mathbf{H}\}$.
3. Joint learning, where we analyze whether private agents and the policymaker can learn $\{\mathbf{H}, \mathbf{F}, \mathbf{V}\}$ jointly.

Preliminaries To place the three learning problems in a unified framework, let us denote by $\mathbf{\Phi}$ the object(s) to be learned. Thus, in the case where only private agents are learning $\mathbf{\Phi} = \{\mathbf{H}\}$. Then, to determine whether $\mathbf{\Phi}$ is learnable we construct and analyze the $\mathbb{T}$-map that relates a perception of $\mathbf{\Phi}$, denoted $\overline{\mathbf{\Phi}}$, to an actual $\mathbf{\Phi}$, $\mathbf{\Phi} = \mathbb{T}\left(\overline{\mathbf{\Phi}}\right)$.

Definition 7 *A fix-point, $\mathbf{\Phi}^*$, of the $\mathbb{T}$-map, $\mathbf{\Phi} = \mathbb{T}\left(\overline{\mathbf{\Phi}}\right)$, is said to be IE-stable if*

$$\lim_{k\uparrow\infty} \mathbb{T}^k\left(\overline{\mathbf{\Phi}}\right) = \mathbf{\Phi}^*,$$

for all $\overline{\mathbf{\Phi}} \neq \mathbf{\Phi}^*$.

It follows that $\mathbf{\Phi}^*$ is IE-stable if and only if it is a stable fix-point of the difference equation

$$\mathbf{\Phi}_{k+1} = \mathbb{T}\left(\mathbf{\Phi}_k\right), \tag{60}$$

where index k denotes the step of the updating process. Similarly,

Definition 8 *A fix-point, $\mathbf{\Phi}^*$, of the $\mathbb{T}$-map, $\mathbf{\Phi} = \mathbb{T}\left(\overline{\mathbf{\Phi}}\right)$, is said to be locally IE-stable if*

$$\lim_{k\uparrow\infty} \mathbb{T}^k\left(\overline{\mathbf{\Phi}}\right) = \mathbf{\Phi}^*,$$

for all $\overline{\mathbf{\Phi}}$ about a neighborhood of $\mathbf{\Phi}^*$.

Let the derivative of the $\mathbb{T}$-map be denoted $\mathbb{DT}(\mathbf{\Phi}^*)$, then it is straightforward to prove the following Lemma.

Lemma 1 *Assume that the derivative map, $\mathbb{DT}(\mathbf{\Phi}^*)$, has no eigenvalues with modulus equal to 1. A fix-point, $\mathbf{\Phi}^*$, of the $\mathbb{T}$-map, $\mathbf{\Phi} = \mathbb{T}\left(\overline{\mathbf{\Phi}}\right)$, is locally IE-stable if and only if all eigenvalues of the derivative map, $\mathbb{DT}(\mathbf{\Phi}^*)$, have modulus less than 1.*

Proof. Following Evans (1985), to analyze the local stability of equation (60) we linearize the equation about $\mathbf{\Phi}^*$. Using matrix calculus results from Magnus and Neudecker (1999, Ch.9), we obtain

$$d\left(vec\left(\mathbf{\Phi}_{k+1}\right)\right) = \mathbb{DT}\left(\mathbf{\Phi}^*\right) d\left(vec\left(\mathbf{\Phi}_k\right)\right)$$

where $\mathbb{DT}(\mathbf{\Phi}^*) = \partial\left(vec\left(\mathbb{T}\left(\mathbf{\Phi}^*\right)\right)\right)\partial\left(vec\left(\mathbf{\Phi}\right)\right)'$. Applying standard results for linear difference equations, if all of the eigenvalues of $\mathbb{DT}(\mathbf{\Phi}^*)$ have modulus less than one, then $\mathbf{\Phi}^*$ is locally stable. In contrast, if one or more of the eigenvalues of $\mathbb{DT}(\mathbf{\Phi}^*)$ have modulus greater than one, then $\mathbf{\Phi}^*$ is not locally stable. ■

Learning by private agents We begin with the case in which only private agents are learning and examine whether private agents can learn $\mathbf{H}$, given $\{\mathbf{F}, \mathbf{V}\}$. For a given policy rule, $\mathbf{u}_t = \mathbf{F}\mathbf{x}_t$, and a postulated private sector decision rule

$$\mathbf{y}_t = \overline{\mathbf{H}}\mathbf{x}_t,$$

the actual private sector decision rule takes the form

$$\mathbf{y}_t = \mathbf{H}\mathbf{x}_t.$$

The perceived low-of-motion will be consistent with a rational expectations equilibrium if it is supported by the evolution of the economy. This yields

$$\mathbf{H} = \left(\overline{\mathbf{H}}\mathbf{A}_{12} - \mathbf{A}_{22}\right)^{-1}\left[\mathbf{A}_{21} + \mathbf{B}_2\mathbf{F} - \overline{\mathbf{H}}\left(\mathbf{A}_{11} + \mathbf{B}_1\mathbf{F}\right)\right]. \tag{61}$$

Equation (61) describes the $\mathbb{T}$-map, $T(\overline{\mathbf{H}})$, from $\overline{\mathbf{H}}$ to $\mathbf{H}$; it is, of course, equivalent to equation (56).

Lemma 2 *A symmetric Markov-perfect Stackelberg-Nash equilibrium is locally IE-stable under private sector learning if and only if all eigenvalues of*

$$-\left[\mathbf{I} \otimes \left(\mathbf{H}\mathbf{A}_{12} - \mathbf{A}_{22}\right)\right]^{-1}\left[\left(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H} + \mathbf{B}_1\mathbf{F}\right)' \otimes \mathbf{I}\right]$$

have modulus less than 1.

Proof. Applying standard matrix calculus rules to equation (61), the total differential can be written as

$$(\mathbf{H}\mathbf{A}_{12}-\mathbf{A}_{22})\,d(\mathbf{H})+d\left(\overline{\mathbf{H}}\right)\mathbf{A}_{12}\mathbf{H}+d\left(\overline{\mathbf{H}}\right)(\mathbf{A}_{11}+\mathbf{B}_1\mathbf{F})=\mathbf{0},$$

which after vectorizing can be rearranged to give

$$vec\,[d(\mathbf{H})]=-\left[\mathbf{I}\otimes(\mathbf{H}\mathbf{A}_{12}-\mathbf{A}_{22})\right]^{-1}\left[(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{H}+\mathbf{B}_1\mathbf{F})'\otimes\mathbf{I}\right]vec\left[d\left(\overline{\mathbf{H}}\right)\right].$$

We apply Lemma 1 to obtain the required result. Note that invertability of $(\mathbf{H}\mathbf{A}_{12}-\mathbf{A}_{22})$ is virtually ensured by the assumption that $\mathbf{A}_{22}$ has full rank. ■

Because the eigenvalues of $\mathbf{M}=\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{H}+\mathbf{B}_1\mathbf{F}$ are all strictly less than $\beta^{-\frac{1}{2}}$, equilibria that are not locally IE-stable under private sector learning are those for which $(\mathbf{H}\mathbf{A}_{12}-\mathbf{A}_{22})$ is close to equaling the null matrix.

Learning by the leader We now turn to the case where the policymaker is learning, but private agents are not. Here we examine whether the policymaker can learn $\{\mathbf{F},\mathbf{V}\}$, given $\{\mathbf{H}\}$. We show that although learning by policymakers is interesting and important in many contexts, here this local IE-stability criterion cannot discriminate among equilibria.

For a given private sector decision rule, $\mathbf{y}_t=\mathbf{H}\mathbf{x}_t$, and a postulated policy rule

$$\mathbf{u}_t=\overline{\mathbf{F}}\mathbf{x}_t,$$

and a postulated value function matrix $\overline{\mathbf{V}}$, the T-map $T(\overline{\mathbf{F}},\overline{\mathbf{V}})$ from $\left\{\overline{\mathbf{F}},\overline{\mathbf{V}}\right\}$ to $\{\mathbf{F},\mathbf{V}\}$, consistent with implementing the best response to the private sector's reaction function, is described by the following updating relationships

$$\begin{aligned}\mathbf{F} &= -\left(\widehat{\mathbf{Q}}+\beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{B}}\right)^{-1}\left(\widehat{\mathbf{U}}'+\beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{A}}\right), &(62)\\ \mathbf{V} &= \widehat{\mathbf{W}}+2\widehat{\mathbf{U}}\mathbf{F}+\mathbf{F}'\widehat{\mathbf{Q}}\mathbf{F}+\beta\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right)'\overline{\mathbf{V}}\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right), &(63)\end{aligned}$$

where $\widehat{\mathbf{W}}$, $\widehat{\mathbf{U}}$, $\widehat{\mathbf{Q}}$, $\widehat{\mathbf{A}}$, and $\widehat{\mathbf{B}}$ are defined by equations (50)—(54) where

$$\begin{aligned}\mathbf{J} &= (\mathbf{A}_{22}-\mathbf{H}\mathbf{A}_{12})^{-1}(\mathbf{H}\mathbf{A}_{11}-\mathbf{A}_{21}),\\ \mathbf{K} &= (\mathbf{A}_{22}-\mathbf{H}\mathbf{A}_{12})^{-1}(\mathbf{H}\mathbf{B}_1-\mathbf{B}_2).\end{aligned}$$

so that they do not depend on $\mathbf{F}$ or $\mathbf{V}$ (or on $\overline{\mathbf{F}}$ or $\overline{\mathbf{V}}$). Notice, that $\mathbf{F}$, given $\mathbf{H}$, is uniquely determined by $\mathbf{V}$, so the key to learning $\mathbf{F}$ is to learn $\mathbf{V}$. As a consequence, without loss of generality we can substitute equation (62) into equation (63) and analyze the learning problem using the concentrated T-map $T(\overline{\mathbf{V}})=\mathbf{V}$.

Lemma 3 *All symmetric Markov-perfect Stackelberg-Nash equilibria are locally IE-stable under policymaker learning.*

Proof. Applying standard matrix calculus rules to equations (62) and (63), total differentials are given by

$$\left(\widehat{\mathbf{Q}}+\beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right)d\left(\mathbf{F}\right)+\beta\widehat{\mathbf{B}}'d\left(\overline{\mathbf{V}}\right)\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right) = \mathbf{0}, \quad (64)$$

$$2\left[\widehat{\mathbf{U}}+\mathbf{F}'\widehat{\mathbf{Q}}+\beta\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right)'\mathbf{V}\widehat{\mathbf{B}}\right]d\left(\mathbf{F}\right)+\beta\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right)'d\left(\overline{\mathbf{V}}\right)\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right) = \mathbf{I}d\left(\mathbf{V}\right). \quad (65)$$

Using equation (64) to solve for $d(\mathbf{F})$ and substituting the resulting expression into equation (65) yields, upon rearranging,

$$\beta\left[-2\left(\widehat{\mathbf{U}}+\beta\widehat{\mathbf{A}}'\mathbf{V}\mathbf{B}\right)\left(\widehat{\mathbf{Q}}+\beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right)^{-1}\widehat{\mathbf{B}}'-2\mathbf{F}'\widehat{\mathbf{B}}'+\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right)'\right]d\left(\overline{\mathbf{V}}\right)\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right)=\mathbf{I}d\left(\mathbf{V}\right),$$

which, given equation (62), collapses to

$$\beta\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right)'d\left(\overline{\mathbf{V}}\right)\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right)=\mathbf{I}d\left(\mathbf{V}\right). \quad (66)$$

After vectorizing and recognizing that $\mathbf{M}=\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}$, equation (66) can be written as

$$vec\left[d\left(\mathbf{V}\right)\right]=\beta\left(\mathbf{M}'\otimes\mathbf{M}'\right)vec\left[d\left(\overline{\mathbf{V}}\right)\right].$$

The matrix $\beta\left(\mathbf{M}'\otimes\mathbf{M}'\right)$ defines the derivative map $\mathbb{DT}(\mathbf{V})$. Applying Lemma 1, a symmetric Markov-perfect Stackelberg-Nash equilibria $\{\mathbf{H},\mathbf{F},\mathbf{M},\mathbf{V},d\}$ is a local IE-stable policy equilibrium if and only if all of the eigenvalues of $\mathbb{DT}(\mathbf{V})$ have modulus less than 1. Because the eigenvalues of $\mathbf{M}$ all have modulus less than $\beta^{-\frac{1}{2}}$ in all symmetric Markov-perfect Stackelberg-Nash equilibria the result follows. ■

Joint learning Finally, we analyze the case in which both private agents and the policymaker are learning. The postulated policy and decision rules are

$$\begin{aligned}\mathbf{y}_t &= \overline{\mathbf{H}}\mathbf{x}_t,\\ \mathbf{u}_t &= \overline{\mathbf{F}}\mathbf{x}_t,\end{aligned}$$

and the postulated value function matrix is $\overline{\mathbf{V}}$. Then the actual policy and decision rules, which are consistent with evolution of the economy and with implementing the best response to the private sector's reaction function, are given by

$$\mathbf{H} = \mathbf{J}+\mathbf{K}\mathbf{F}, \quad (67)$$

$$\mathbf{F} = -\left(\widehat{\mathbf{Q}}+\beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{B}}\right)^{-1}\left(\widehat{\mathbf{U}}+\beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{A}}\right), \quad (68)$$

$$\mathbf{V} = \widehat{\mathbf{W}}+2\widehat{\mathbf{U}}\mathbf{F}+\mathbf{F}'\widehat{\mathbf{Q}}\mathbf{F}+\beta\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right)'\overline{\mathbf{V}}\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right), \quad (69)$$

where

$$\mathbf{J} = \left(\mathbf{A}_{22}-\overline{\mathbf{H}}\mathbf{A}_{12}\right)^{-1}\left(\overline{\mathbf{H}}\mathbf{A}_{11}-\mathbf{A}_{21}\right), \quad (70)$$

$$\mathbf{K} = \left(\mathbf{A}_{22}-\overline{\mathbf{H}}\mathbf{A}_{12}\right)^{-1}\left(\overline{\mathbf{H}}\mathbf{B}_{1}-\mathbf{B}_{2}\right), \quad (71)$$

and $\widehat{\mathbf{W}}$, $\widehat{\mathbf{U}}$, $\widehat{\mathbf{Q}}$, $\widehat{\mathbf{A}}$, and $\widehat{\mathbf{B}}$ are defined by equations (50)—(54) and are functions of $\mathbf{J}$ and $\mathbf{K}$.

Given equations (70) and (71), equations (67)—(69) describe the $\mathbb{T}$-map, $T\left(\overline{\mathbf{H}}, \overline{\mathbf{F}}, \overline{\mathbf{V}}\right)$, from $\left\{\overline{\mathbf{H}}, \overline{\mathbf{F}}, \overline{\mathbf{V}}\right\}$, to $\{\mathbf{H}, \mathbf{F}, \mathbf{V}\}$.

Lemma 4 *A symmetric Markov-perfect Stackelberg-Nash equilibrium is locally IE-stable under joint learning if and only if all eigenvalues of the matrix* $\mathbf{P}^{-1}\mathbf{L}$ *in*

$$vec\left[d\left(\mathbf{G}\right)\right] = \mathbf{P}^{-1}\ \mathbf{L} vec\left[d\left(\overline{\mathbf{G}}\right)\right],$$

where $vec\left[d\left(\mathbf{G}\right)\right] = \left[\begin{array}{ccc} vec\left[d\left(\mathbf{H}\right)\right]^{'} & vec\left[d\left(\mathbf{F}\right)\right]^{'} & vec\left[d\left(\mathbf{V}\right)\right]^{'} \end{array}\right]^{'}$ *and* $\mathbf{P}$ *and* $\mathbf{L}$ *are characterized below, have modulus less than* 1*.*

Proof. Total differentials of equations (67)—(71) about the point $\{\mathbf{H}, \mathbf{F}, \mathbf{V}, \mathbf{J}, \mathbf{K}\}$ are given by

$$\begin{aligned}
\mathbf{0} &= d\left(\mathbf{J}\right) + d\left(\mathbf{K}\right)\mathbf{F} + \mathbf{K}d\left(\mathbf{F}\right) - d\left(\mathbf{H}\right), && (72)\\
\mathbf{0} &= d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{A}} - \left(\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12}\right)d\left(\mathbf{J}\right), && (73)\\
\mathbf{0} &= d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{B}} - \left(\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12}\right)d\left(\mathbf{K}\right), && (74)\\
\mathbf{0} &= \beta\widehat{\mathbf{B}}^{'}d\left(\overline{\mathbf{V}}\right)\mathbf{M} + \left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}^{'}\mathbf{V}\widehat{\mathbf{B}}\right)d\left(\mathbf{F}\right) + 2\left(\mathbf{K}^{'}\mathbf{W}_{22} + \mathbf{U}_{2}^{'} + \beta\widehat{\mathbf{B}}^{'}\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{K}\right)\mathbf{F}\\
&\quad + \left(\mathbf{W}_{12} + \mathbf{J}^{'}\mathbf{W}_{22} + \beta\widehat{\mathbf{A}}^{'}\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{K}\right) + \left(\mathbf{K}^{'}\mathbf{W}_{22} + \mathbf{U}_{2}^{'} + \beta\widehat{\mathbf{B}}^{'}\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{J}\right), && (75)\\
\mathbf{0} &= 2\left(\widehat{\mathbf{U}} + \mathbf{F}^{'}\widehat{\mathbf{Q}} + \beta\mathbf{M}^{'}\mathbf{V}\widehat{\mathbf{B}}\right)d\left(\mathbf{F}\right) + 2\left(\mathbf{W}_{12} + \mathbf{H}^{'}\mathbf{W}_{22} + \mathbf{F}^{'}\mathbf{U}_{2}^{'} + \beta\mathbf{M}^{'}\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{J}\right)\\
&\quad + 2\left(\mathbf{W}_{12} + \mathbf{H}^{'}\mathbf{W}_{22} + \mathbf{F}^{'}\mathbf{U}_{2}^{'} + \beta\mathbf{M}^{'}\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{K}\right)\mathbf{F} + \beta\mathbf{M}^{'}d\left(\overline{\mathbf{V}}\right)\mathbf{M} - d\left(\mathbf{V}\right). && (76)
\end{aligned}$$

Now, using equations (73) and (74) to solve for $d\left(\mathbf{J}\right)$ and $d\left(\mathbf{K}\right)$, respectively, and substituting these expressions into equations (72), (75), and (76) produces

$$\begin{aligned}
\mathbf{0} &= \mathbf{K}d\left(\mathbf{F}\right) + \left(\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12}\right)^{-1}d\left(\overline{\mathbf{H}}\right)\mathbf{M} - d\left(\mathbf{H}\right), && (77)\\
\mathbf{0} &= \beta\widehat{\mathbf{B}}^{'}d\left(\overline{\mathbf{V}}\right)\mathbf{M} + \left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}^{'}\mathbf{V}\widehat{\mathbf{B}}\right)d\left(\mathbf{F}\right)\\
&\quad + \left(\mathbf{W}_{12} + \mathbf{J}^{'}\mathbf{W}_{22} + \beta\widehat{\mathbf{A}}^{'}\mathbf{V}\mathbf{A}_{12}\right)\left(\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12}\right)^{-1}d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{B}}\\
&\quad + 2\left(\mathbf{K}^{'}\mathbf{W}_{22} + \mathbf{U}_{2}^{'} + \beta\widehat{\mathbf{B}}^{'}\mathbf{V}\mathbf{A}_{12}\right)\left(\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12}\right)^{-1}d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{B}}\mathbf{F}\\
&\quad + \left(\mathbf{K}^{'}\mathbf{W}_{22} + \mathbf{U}_{2}^{'} + \beta\widehat{\mathbf{B}}^{'}\mathbf{V}\mathbf{A}_{12}\right)\left(\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12}\right)^{-1}d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{A}} && (78)\\
\mathbf{0} &= 2\left(\widehat{\mathbf{U}} + \mathbf{F}^{'}\widehat{\mathbf{Q}} + \beta\mathbf{M}^{'}\mathbf{V}\widehat{\mathbf{B}}\right)d\left(\mathbf{F}\right) + \beta\mathbf{M}^{'}d\left(\overline{\mathbf{V}}\right)\mathbf{M} - d\left(\mathbf{V}\right)\\
&\quad + 2\left(\mathbf{W}_{12} + \mathbf{H}^{'}\mathbf{W}_{22} + \mathbf{F}^{'}\mathbf{U}_{2}^{'} + \beta\mathbf{M}^{'}\mathbf{V}\mathbf{A}_{12}\right)\left(\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12}\right)^{-1}d\left(\overline{\mathbf{H}}\right)\mathbf{M}, && (79)
\end{aligned}$$

where, again, the invertability of $\left(\mathbf{A}_{22} - \mathbf{H}\mathbf{A}_{12}\right)$ is virtually ensured by the assumption that $\mathbf{A}_{22}$ has full rank. By vectorizing and stacking equations (77)—(79) they can be written in the form

$$\mathbf{P} vec\left[d\left(\mathbf{G}\right)\right] = \mathbf{L} vec\left[d\left(\overline{\mathbf{G}}\right)\right],$$

where

$$\mathbf{P}=\left[\begin{array}{ccc}\mathbf{I} & -\mathbf{K} & \mathbf{0}\\ \mathbf{0} & -\left(\widehat{\mathbf{Q}}+\beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right) & \mathbf{0}\\ \mathbf{0} & -2\left(\widehat{\mathbf{U}}+\mathbf{F}'\widehat{\mathbf{Q}}+\beta\mathbf{M}'\mathbf{V}\widehat{\mathbf{B}}\right) & \mathbf{I}\end{array}\right],$$

and $\mathbf{L}$ is defined implicitly by equations (77)—(79). Because $\left(\widehat{\mathbf{Q}}+\beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right)$ has full rank in any symmetric Markov-perfect Stackelberg-Nash equilibrium, $\mathbf{P}$ too has full rank. The result follows. ■

Lemma 5 *The equilibrium identified by Oudiz and Sachs (1985) and all equilibria identified by Backus and Driffill (1986) are IE-stable under joint learning.*

Proof. The iterative numerical schemes employed by the Backus and Driffill (1986) and Oudiz and Sachs (1985) solution methods coincide with the learning scheme described by the $\mathbb{T}$-map (67)—(69). As a consequence, these numerical solution methods apply direct numerical iterations on the non-linear T-map. If these numerical solution methods converge to a fix-point, then, by construction, the resulting equilibrium is IE-stable under joint learning. ■

A.2.2 Self-enforceability

We now approach the coordination problem by asking whether an equilibrium is self-enforceable (Bernheim et al. (1987), Bernheim and Whinston (1987)), robust to the potential formation of non-cooperative coalitions. Assume that the model has N symmetric Markov-perfect Stackelberg-Nash equilibria. Because the economic environment is one in which there is complete and perfect information, the existence and nature of all N equilibria is known to all agents. Moreover, the N equilibria can (invariably) be welfare ranked and, as a consequence, agents are not indifferent to which equilibrium prevails.

Treating the policy rules associated with the N equilibria as a set of policy actions, because the equilibria are Nash, if policymakers in periods $s=t+1,...,\infty$ are expected to play $\overline{\mathbf{F}}_j$, $j=1,...,N$, then the period-t policymaker's best response is to also play $\mathbf{F}_j$. However, although it is never beneficial for the period-t policymaker to unilaterally deviate from Nash play, the period-t policymaker can potentially benefit from deviations that involve multiple policymakers. With this in mind, we introduce the possibility that a "small" coalition of policymakers could form that may deviate from the play prescribed in equilibrium j. The coalitions that we envisage are motivated by the fact that policymakers have tenures spanning multiple decision periods and, as a consequence, we model them in terms of sequential players.

Let (p_j+1) represent the number of sequential players in a potential coalition and consider the period-t policymaker's best response where the predicted future play is given by $\{\mathbf{F}_i^{t+1}$, ..., $\mathbf{F}_i^{t+p_j}$, $\mathbf{F}_j^{t+p_j+1}$, $\mathbf{F}_j^{t+p_j+2}$, ...$\}$, $j\neq i$, with private agents in periods $s=t,...,\infty$ responding according to their reaction function. In this scenario, during periods $s=t+p_j+1,...,\infty$ the policy rule and private-sector decision rules are given by $\mathbf{F}_j$ and $\mathbf{H}_j$, respectively. However, during periods $s=t,...,t+p_j$ the policy rule is given by $\mathbf{F}_i$ and private agents respond according to their reaction function,

$$\mathbf{H}^s = \left(\mathbf{H}^{s+1}\mathbf{A}_{12} - \mathbf{A}_{22}\right)^{-1} \left[\mathbf{A}_{21} + \mathbf{B}_2\mathbf{F}_i - \mathbf{H}^{s+1}\left(\mathbf{A}_{11} + \mathbf{B}_1\mathbf{F}_i\right)\right]. \tag{80}$$

Given equation (80), the law-of-motion for the state vector during periods $s = t, ..., t + p_j$ is

$$\mathbf{M}^s = \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H}^s + \mathbf{B}_1\mathbf{F}_i.$$

We know that if $p_j = 0$, then the period-t policymaker's best response is to play $\mathbf{F}_j$. However, as p_j increases, the period-t policymaker's best response can switch from $\mathbf{F}_j$ to $\mathbf{F}_i$. For each $\mathbf{F}_j$, we calculate the number of periods of multilateral deviation p_j required to switch the period-t policymaker's best response from $\mathbf{F}_j$ to $\mathbf{F}_i$. Of course, although the period-t policymaker's best response may switch from $\mathbf{F}_j$ to $\mathbf{F}_i$ as p_j increases, it need not. In fact, whether the period-t policymaker's best response switches from $\mathbf{F}_j$ to $\mathbf{F}_i$ as p_j increases turns on whether equilibrium i is Pareto-preferred to equilibrium j and on whether equilibrium i is locally IE-stable under private sector learning.

Lemma 6 *The period-t policymakers best response will switch from $\mathbf{F}_j$ to $\mathbf{F}_i$ in the limit as $p_j \uparrow \infty$ if and only if equilibrium i is Pareto-preferred to equilibrium j and equilibrium i is locally IE-stable under private sector learning.*

Proof. Consider equation (80). If equilibrium i is locally IE-stable under private sector learning, then, $\mathbf{H}^s \to \mathbf{H}^i$ in the limit as $p_j \uparrow \infty$, which implies $\mathbf{M}^s \to \mathbf{M}^i$ and $\mathbf{V}^s \to \mathbf{V}^i$. Because equilibrium i Pareto-dominates equilibrium j, the period-t policymaker's best response must switch from $\mathbf{F}_j$ to $\mathbf{F}_i$. On the contrary, if equilibrium i is not locally IE-stable under private sector learning, then although $\mathbf{H}^s$ may converge to $\widetilde{\mathbf{H}} \neq \mathbf{H}^i$ in the limit as $p_j \uparrow \infty$, because $\widetilde{\mathbf{H}} \neq \mathbf{H}^i$ the period-t policymaker's best response cannot be $\mathbf{F}_i$. ■

An additional issue that we consider is whether coalition forming can generate a switch from the prevailing equilibrium to the Pareto-preferred equilibrium and, if so, how large of a coalition is required to generate such a switch. It follows from Lemma 6 that the Pareto-preferred equilibrium must be locally IE-stable under private sector learning if such a switch is to occur.

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