Bargaining and Investment*

by

Francesca Flamini

University of Glasgow

Department of Economics

Adam Smith Building

Glasgow, G12 8RT, UK

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Abstract

The focus of this paper is on repeated bargaining games in which two parties can decide how much to invest and how to share the remaining surplus for their own consumption. The game is dynamic since the current level of investment affects future surpluses. We characterise an MPE without delays in general terms and show the parametrical effects for the specific case in which parties share the surplus equally. We show that the relatively more patient player invests more than his opponent, for a given capital stock. Moreover, if the probability of becoming a proposer decreases for the more patient player, then such a player reduces his investment, while the relatively impatient player increases his investment.

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1 Introduction

This paper focuses on bargaining games in which parties can invest part of the surplus. Moreover, the invested surplus affects the size of future surpluses. Bargaining games which allow parties to make both investment and consumption decisions and for a repeated number of times are almost unexplored. There are two main strands of the literature, namely, on the *hold-up* problem and on the *tragedy of the commons,* which are related to the problem considered in this paper. However, there are fundamental differences between these models and our dynamic bargaining game with investment. In the *hold-up* problem bargaining and investment are two completely separate processes (see, for instance, Gibbons (1992), Muthoo (1996), Gul (2001)). The investment takes place before the bargaining stage and only one party is involved (he bears all the costs), moreover, the investment is once for all. Parties then can share a surplus whose size depends on the initial investment. Differently, in this paper the focus is on parties who *jointly* and *repeatedly* need to decide how much to invest and consume. We assume that they make their consumption and investment decisions at the same time but this assumption is not crucial (given the dynamic structure of the game, their decisions would be linked even if the consumption decisions were taken before or after the investment plan). The second strand of the literature, on the *tragedy of the commons*, considers different parties who can extract part of the surplus for their own consumption and the remaining surplus will affect the size available in the next period (see, for instance, Levhari and Mirman, 1980, Dutta and Sandaram, 1993). Although this is a dynamic accumulation game, there is no bargaining, in particular everyone can consume as much as he wishes (as long as the sums of all the demands is smaller than the size of the surplus).

The model closer to ours, with focus on a repeated bargaining game with investment decisions in addition to the standard consumption decisions, is Muthoo (1999). However, the most important difference with our paper is that in Muthoo (1999) the focus is on steady-state stationary subgame perfect equilibria, while ours is on Markov Perfect Equilbria (MPE). This implies that in the former, the investment decisions are strongly simplified since parties need to investment as much as it is necessary so as to have surpluses of the same size. Indeed Muthoo's aim is to apply his infinitely repeated game where parties share an infinite number of cakes with the same size (Muthoo, 1995). In this sense the problem of how much parties should invest remains open. A first step to address this question is in another paper of ours (Flamini, 2002), in which we assume that parties can take their consumption and investment decisions under the assumption of a take-it-or-leave bargaining structure. Although this assumption is necessary to deal with a dynamic game, that is otherwise very complex, it is not always realistic, since one can imagine situations where following a rejection of an offer, a counter-proposal can take place. To address this issue in this paper we allow players to continue the bargaining process after a rejection. Although this case is very complex (since the game can be represented as a recursive problem with a constraint that embodies another recursive problem), we are able to characterise the MPE without delay. While the indifference conditions (which are conditions that says that a proposal makes a player indifferent between accepting and rejecting such a proposal) are important instrument for the definition of an equilibrium without delay in bargaining games, we show that these work only in rare conditions in our dynamic bargaining game. To show the parametrical effects on the equilibrium outcome, players are assumed to share equally the surplus not invested. Although also in this case the analytical solution cannot be fully derived, by numerical evaluations, we show that the more patient player invests more than his opponent for a given level of capital stock. Moreover, if the more patient player becomes less likely to make a proposal, then his investment decreases while his opponent's investment increases for a given capital stock.

The paper is organised as follows. In the next section we present the model, in section 3 we characterised an MPE without delays. To analyse the equilibrium in more detail we consider the case in which parties split the surplus not invested equally, in section 4. Some final remarks conclude the paper.

2 The model

We consider a two-player bargaining game in which bargaining and production stages alternate (and each stage can start only after the other has taken place). At the production stage, a surplus is generated according to the production function $F(k_t) = k_t$, where k_t is the capital stock at period t, with $t = 0, 1, ...$ Production takes place in an interval of time τ. Once the output is generated, $F(k_t)$, the bargaining stage begins and players attempt to divide $F(k_t)$. The bargaining stage is a classic random-proposer procedure in which each party can become a proposer with a positive probability. In general, in period t player 1 (2, respectively) can become a proposer with probability p $(1 - p$, respectively), with $0 < p < 1$. A proposal by player i is a pair $({}_ix_t, {}_iI_t)$, where ${}_iI_t$ is the investment level proposed by i at time t and ${}_ix_t$ is the share demanded by i over the remaining surplus at time t. The proposal $({}_ix_t, {}_iI_t)$ depends on capital, denoted by k_t, that is the state variable in the model. Our notation is simplified, in the sense that the subscript t in the proposal indicates that this is conditional on the capital stock at t, k_t. If there is an acceptance, the bargaining stage ends and the proposer's current per-period utility is,

$$u_i({}_ix_t, {}_iI_t) = \frac{[{}_ix_t(F(k_t) - \ {}_iI_t)]^{1-\eta}}{1-\eta}$$

The output available at the next bargaining stage (at $t + 1$) is $F(k_{t+1})$, where k_{t+1} is the capital stock in the next period and it is given by the investment level ${}_iI_t$ and the capital remaining after depreciation, $k_{t+1} = {}_iI_t + (1-\lambda)k_t$, where λ is the depreciation rate ($0 < \lambda \leq 1$). If there is a rejection, after an interval of time Δ, a randomly-selected proposer can make an offer. Player i's time preference is represented by his discount rate r_i (with $i = 1, 2$). We take into account the fact that intervals of time have different lengths, with two distinct discount factors: the between-cake discount factor $\alpha_i = exp(-r_i\tau)$ which takes into account that production takes time and the between-cake discount factor $\delta_i = exp(-r_i\Delta)$ that takes into account that there is an interval of time between a rejection and a new proposal. In the first period, at $t = 0$, a bargaining stage starts and the surplus available is 1, by assumption.

3 Characterisation of an MPE without delay

In this section we attempt to characterise an MPE without delay, if it exists. Some remarks on the existence of MPE with delay are in section 5. Let $v_i(k_t)$ be the expected discounted utility that player i attempts to maximise as a proposer, while $w_j(k_t)$ is the expected discounted utility of player j when he accepts player i's proposal $({}_ix_t, {}_iI_t)$, i.e.,

$$v_i(k_t) = \frac{[{}_i x_t (k_t - {}_i I_t)]^{1-\eta}}{1-\eta} +$$

$$+\sum_{p=1}^{\infty}\sum_{s=2^{p-1}}^{2^p-1}\Big(\prod_{h \in h^{t+s} \backslash h^t} h\Big)\alpha_i^{t+s}\Big[p_i \frac{[{}_i x_{t+s}(k_{t+s}^{h^{t+s}} - {}_i I_{t+s})]^{1-\eta}}{1-\eta} + (1-p_i)\frac{[(1-{}_j x_{t+s})(k_{t+s}^{h^{t+s}} - {}_j I_{t+s})]^{1-\eta}}{1-\eta}\Big]$$

$$w_j(k_t) = \frac{[(1-{}_i x_t)(k_t - {}_i I_t)]^{1-\eta}}{1-\eta} +$$

$$+\sum_{p=1}^{\infty}\sum_{s=2^{p-1}}^{2^p-1}\Big(\prod_{h \in h^{t+s} \backslash h^t} h\Big)\alpha_i^{t+s}\Big[p_i \frac{[(1-{}_i x_{t+s})(k_{t+s}^{h^{t+s}} - {}_i I_{t+s})]^{1-\eta}}{1-\eta} + (1-p_i)\frac{[{}_j x_{t+s}(k_{t+s}^{h^{t+s}} - {}_j I_{t+s})]^{1-\eta}}{1-\eta}\Big]$$

(1)

where each element of a potential history h^t, h, is equal to p_i assuming player i proposes and $(1 - p_i)$ when player j is assumed to propose (except for the unique element of $h^{t+1} \backslash h^t$, that is 1 by assumption) for $t = 1, 2 \dots \infty$. The potential history h^t uniquely indicates the sequence of proposers to reach the node t from 1, where the nodes considered are only the ones in which an offer is to be made. These are numbered sequentially from 1. At each node there are two possibilities either i or j will propose next, in each period the lowest number is given to the node where i will propose next. Accordingly, the product of the elements of h^t gives the probability of reaching node t from node 1.

In its recursive form the problem is as follows

$$
\begin{aligned}
V_i(k_t) &= \max_{{}_i x_t, {}_i I_t} \frac{[{}_i x_t(F(k_t) - \ {}_i I_t)]^{1-\eta}}{1-\eta} + \alpha_i(p_i V_i(k_{t+1}) + (1-p_i)W_i(k_{t+1})) \quad (2) \\
s.t. W_j(k_t) &\geq \delta_j[p_j V_j(k_t) + (1-p_j)W_j(k_t)] \quad (3) \\
\text{with } k_{t+1} &= \begin{cases} (1-\lambda)k_t + \ {}_i I_t \text{ if there is an acceptance} \\ k_t \text{ otherwise} \end{cases} \quad (4)
\end{aligned}
$$

where $V_i(k_t)$ is the *value function,* i.e., the optimal expected utility to player i as a proposer, while $W_j(k_t)$ is the optimal expected utility to player j as a responder. The indifference conditions (so that a responder is indifferent between accepting and rejecting a proposal) are important instruments for a solution of a bargaining game. With an ultimatum procedure, these are strong assumptions, since they impose that a responder obtains a null expected utility in case of a rejection ($W_j(k_t)$ for any j and t), but in our model a responder obtains a positive expected payoff in the case he rejects a proposal. Can the indifference conditions hold? In the next section, we show that the answer is yes, but only in specific cases.

3.1 The Indifference Conditions

The indifference conditions in our model are constraints (3) as equalities. Then condition (3) can be written as follows

$$
W_j(k_t) = \frac{\delta_j p_j}{1 - \delta_j(1-p_j)} V_j(k_t) \quad (5)
$$

for any j and k. Then by using (5) in (2) for player i at $t+1$, the problem for a proposer becomes as follows:

$$V_i(k_t) = \max_{{}_ix_t, {}_iI_t} \frac{[{}_ix_t(F(k_t) - {}_iI_t)]^{1-\eta}}{1-\eta} + \beta_i V_i(k_{t+1}) \quad (6)$$

$$k_t = (1-\lambda)k_t + {}_iI_t \quad (7)$$

with $\beta_i = \frac{\alpha_i p_i}{1-\delta_i(1-p_i)}$. This is very similar to a standard growth model in which a social planner has to decide the consumption and investment path. In particular since in problem (6)-(7) there are no strategic interactions, a proposer, just like the social planner, will not waste any resources and therefore ${}_ix_t = 1$. Moreover, the consumption plan ${}_ic_t = {}_ix_t(F(k_t) - {}_iI_t) = F(k_t) - {}_iI_t$ is characterised in the following proposition.

Proposition 1 *The solution to problem (6)-(7) is given by the following consumption plan*

$${}_ic_t = k_t(2-\lambda)(1-(\beta_i(2-\lambda)^{1-\eta})^{1/\eta}) \quad (8)$$

with $\beta_i(2-\lambda) < 1$. *This can be a solution to the bargaining (2)-(4) if*

$$\alpha_j^{1/\eta}(2-\lambda)^{\frac{1-\eta}{\eta}}(2-\delta_j)^{2\eta-1} = \delta_j[2 - \alpha_j\left(\frac{2-\lambda}{2-\delta_j}\right)^{\frac{1-\eta}{\eta}} \quad (9)$$

for $i, j = 1, 2$ *and* $i \neq j$.

Proof. The proof to show that the solution of (6)-(7) is given by ${}_ic_t$ in (8) is standard and therefore omitted. The condition (9) is obtained by using the solution ${}_ic_t$, (8), in the immediate-agreement condition (5). ■

Condition (9) implies that the relationships among the parameters are very specific. In conclusion, the indifference conditions, which are important instruments to derive equilibrium outcomes of bargaining games can be used in our dynamic bargaining game only under some specific cases, that is, when (9) holds.

3.2 A General Solution

In this section we consider the explicit form of problem (2)-(4), i.e., by using (1), and we derive the necessary conditions for an MPE without delays. First of all, the FOC for the proposer problem are as follows:

$$\begin{aligned} (1- {}_ix_t) &= {}_ix_t d_j^{1/\eta} && (10)\\ \frac{{}_ix_t^{1-\eta} + d_j(1- {}_ix_t)^{1-\eta}}{{}_iz_t} &= \frac{\alpha_i p_i\, {}_ix_{t+1}^{1-\eta} + \alpha_j d_j p_i (1- {}_ix_{t+1})^{1-\eta}}{{}_iz_{t+1}} + && (11)\\ &\quad + \frac{\alpha_i(1-p_i)(1- {}_jx_{t+1})^{1-\eta} + \alpha_j d_j p_j\, {}_jx_{t+1}^{1-\eta}}{{}_jz_{t+1}} \end{aligned}$$

where $d_j = {}_im_t(1-\delta_j(1-p_j))$ and ${}_iz_t = (k_t - {}_iI_t)^\eta$ where ${}_im_t$ is the Kuhn-Tucker multiplier, for any $i, j = 1, 2$ with $i \neq j$ and $t = 0, 1, 2, ...\infty$. Second, an additional condition for the solution of the proposer's problem is the transversality condition, that is, at the limit as t tends to zero, the utility value of the discounted capital stock $(\alpha_i^t k_t v_i'(k_t))$ goes to zero.

By using (10), the explicit solution of ${}_ix_t$ is as follows:

$${}_ix_t = \frac{1}{1 + [{}_im_t(1-\delta_j(1-p_j))]^{1/\eta}} \qquad (12)$$

In general, ${}_ix_t$ depends on k_t via ${}_im_t$ and is not larger than 1, since ${}_im_t$ is non-negative. If in equilibrium the immediate-agreement condition (3) holds with strict inequality for player 1 at some t, by the principle of complementary slackness, the multiplier (3) must be zero, which implies that in equilibrium player i demands the entire surplus not invested ${}_ix_t = 1$. On the other hand, if for player i, at some t, the constraint holds with equality then the multiplier is non-negative. As shown in section 3.1, the indifference conditions are strong assumptions since they state that the constraint (3) holds as an equality for any i and t and the multiplier is zero.

In the next section we focus on the case in which each player demands the entire surplus not invested whenever he proposes and, by the principle of complementary slackness, the immediate-agreement conditions hold with strict inequality for any i and t.

3.3 The Ultimatum-like MPE

It is now assumed that the immediate-acceptance conditions (3) hold in general as inequalities and therefore, a proposer is able to consume the whole portion of the surplus not invested. This section is closely related to Flamini (2002), where the focus is on an ultimatum bargaining procedure, which implies that a proposer can ask as much as he wishes without fearing a rejection and therefore in equilibrium he will ask the entire surplus not invested. The main difference with the ultimatum procedure is that in our case the proposer needs to make sure that demanding the entire surplus not invested is an acceptable proposal while in the ultimatum structure the immediate agreement condition is always satisfied "by construction". That is, once the game is solved, we need to find out for which set of parameters the conditions (3) are satisfied. The following proposition and corollary define this set in general terms and under a specific case, respectively.

Proposition 2 *There is an MPE in which each player extracts the entire surplus not invested as in an ultimatum bargaining procedure, if the conditions for the ultimatum procedure hold and, in addition, the discount factor* δ_i *is sufficiently small (so that* ${}_{LB}\psi_i < {}_{UB}\psi_i$ *for any i).*

Proof. According to the *guess and verify* method, first the function of the value function is guessed, but the coefficients are left undetermined, then the guess is verified by showing that there is a unique value of the coefficient that make the guess correct. Our 'guess' is that the value function is a function of the capital stock of the same form as the utility function. Then the players' optimisation problem can be written as follows.

$$\frac{\phi_i k_t^{1-\eta}}{1-\eta} = \max_{{}_ic_t} \frac{{}_ic_t^{1-\eta}}{1-\eta} + \alpha_i \underline{\beta}_i \frac{k_{t+1}^{1-\eta}}{1-\eta} \tag{13}$$

$$\underline{\beta}_i = p_i\phi_i + (1-p_i)\mu_i \tag{14}$$

$$\frac{\mu_i k_t^{1-\eta}}{1-\eta} = 0 + \alpha_i \underline{\beta}_i \frac{(\varphi_i k_t)^{1-\eta}}{1-\eta} \tag{15}$$

$$k_{t+1} = k_t(2-\lambda) -_i c_t \text{ given that there is an acceptance} \tag{16}$$

where ${}_ic_t = {}_ix_t(F(k_t)- {}_iI_t) = F(k_t)- {}_iI_t$. The proof consists in writing the unknown parameters $\phi_i, \underline{\beta}_i$, μ_i, φ_i as a function of auxiliary variables ψ_i and ψ_j. In particular after some manipulations (see Flamini (2002)) for more details), these are as follows.

$$\underline{\beta}_i = \psi_i\phi_i \tag{17}$$

$$\phi_i = \frac{(2-\lambda)^{1-\eta}}{\left(1-(2-\lambda)^{\frac{1-\eta}{\eta}}\alpha_i^{\frac{1}{\eta}}\psi_i^{\frac{1}{\eta}}\right)^{\eta}} \tag{18}$$

$$\mu_i = \frac{(\psi_i - p_i)\phi_i}{1-p_i} \tag{19}$$

$$\varphi_i = (2-\lambda)^{\frac{1}{\eta}}\psi_i^{\frac{1}{\eta}}\alpha_i^{\frac{1}{\eta}}. \tag{20}$$

where ψ_i and ψ_j are the solution of the following system of two equations:

$$\psi_i = \frac{p_i}{1-(2-\lambda)^{\frac{1-\eta}{\eta}}(1-p_i)\alpha_i^{\frac{1}{\eta}}\psi_j^{\frac{1-\eta}{\eta}}} \tag{21}$$

with $i, j = 1, 2$ and $i \neq j$. If there is a unique solution (ψ_1^*, ψ_2^*), this uniquely defines the unknown parameters and therefore the verification phase ends. If there is more than one solution, or none, then the guess and verifying methods fails to deliver a solution. Next, we define the interval of interest in which only one solution can be found. This interval is defined by the following constraints: first ψ_1^* and ψ_2^* are non negative (so that we obtain a non-negative equilibrium); second, the immediate-agreement condition is satisfied and third, the transversality condition is satisfied. These conditions imply that ψ_j^* has to belong to the interval $[_{LB}\psi_j,_{UB}\psi_j)$ where

$$
\begin{aligned}
{}_{LB}\psi_j &= \max\left\{p_j, \frac{\delta_i^{\frac{1-\eta}{\eta}}}{(2-\lambda)\alpha_j\alpha_i^{\frac{1-\eta}{\eta}}}\right\} && (22)\\
{}_{UB}\psi_j &= \min\left\{{}_d\psi_j, {}_{TC}\psi_j, {}_u\psi_j\right\} && (23)\\
{}_d\psi_j &= 1/a_i && (24)\\
{}_{TC}\psi_j &= \frac{1}{(2-\lambda)\alpha_j\alpha_i^{\frac{1-\eta}{\eta}}} && (25)\\
{}_u\psi_j &= \frac{1}{(2-\lambda)^{1-\eta}\alpha_j} && (26)
\end{aligned}
$$

with $a_i = (2-\lambda)^{\frac{1-\eta}{\eta}}(1-p_i)\alpha_i^{\frac{1}{\eta}}$ and $a_j = (2-\lambda)(1-p_j)^{\frac{1-\eta}{\eta}}\alpha_j^{\frac{1}{1-\eta}}$, $i, j = 1, 2$ and $i \neq j$. In the ultimatum-bargaining procedure ${}_{LB}\psi_j$ is simply p_j since an offer is always accepted. When the immediate-agremeent condition is binding, then ${}_{LB}\psi_j$ is larger then p_j and therefore the interval of interest is smaller. Moreover, if a player is sufficiently patient ($\delta_i \to 1$) ${}_{LB}\psi_j$ can be even larger than the ${}_{UB}\psi_j$ therefore δ_i is required to be sufficiently small so that ${}_{LB}\psi_j < {}_{UB}\psi_j$. Then the proof consists in showing that only one solution can be found in this interval (see Flamini, 2002). ■

To show the result in a more transparent manner, we assume that players are symmetric, that is, $p_i = 0.5$, $r_i = r$ for any i and, in addition, $\eta = 0.5$.

Corollary 3 *For* $p_i = 0.5$, $r_i = r$ *for any* i *and* $\eta = 0.5$, *there in an MPE in which each player extracts the entire surplus not invested as in an ultimatum bargaining procedure, if* $\delta < \alpha(2-\lambda)^{1/2} < 1$.

This implies that only when the depreciation rate is low, this equilibrium is interesting. Indeed, if there is maximum depreciation ($\lambda = 1$), an ultimatum-MPE is sustainable only if the interval of time between a rejection and a new proposal, Δ, is larger than the interval of time between an acceptance an a new proposal, τ (so that $\delta < \alpha$), which is a strong assumption.

4 Splitting the Surplus Equally

To highlight the characteristics of the dynamic bargaining game, in this section we assume that parties split the available surplus equally. For instance, before entering a business two partners sign a contract that specifies that each will obtain half of the profits not re-invested. In this case, the bargaining game remains unsolved, since a proposed investment plan still needs to be accepted by the other party. However, since the consumption plan is exogenously defined then the dynamic game is simplified and can be solved numerically.

In an MPE without delay in which parties share equally a surplus, each proposer will attempt to solve the following recursive problem:

$$V_i(k_t) = \max_{{}_iI_t} \frac{[1/2(k_t - {}_iI_t)]^{1-\eta}}{1-\eta} + \alpha_i(p_i V_i(k_{t+1}) + (1-p_i)W_i(k_{t+1})) \quad (27)$$

$$W_j(k_t) \geq \delta_j[p_j V_j(k_t) + (1-p_j)W_j(k_t)] \quad (28)$$

$$W_j(k_t) = \frac{[1/2(k_t - {}_iI_t)]^{1-\eta}}{1-\eta} + \alpha_j(p_j V_j(k_{t+1}) + (1-p_j)W_j(k_{t+1})) \quad (29)$$

$$k_{t+1} = (1-\lambda)k_t + {}_iI_t \quad (30)$$

By using the guess and verifying method, with the guess that the value function has the same form as the per-period utility function, problem (27)-(30) can re-written as follows

$$\frac{\phi_i k_t^{1-\eta}}{1-\eta} = \max_{{}_iI_t} \frac{[1/2(k_t - {}_iI_t)]^{1-\eta}}{1-\eta} + \alpha_i\beta_i \frac{k_{t+1}^{1-\eta}}{1-\eta} \quad (31)$$

$$\beta_i = p_i\phi_i + (1-p_i)\mu_i \quad (32)$$

$$\frac{\mu_j k_t^{1-\eta}}{1-\eta} \geq \delta_j\alpha_j\beta_j \frac{(\varphi_i k_t)^{1-\eta}}{1-\eta} \quad (33)$$

$$k_{t+1} = k_t(2-\lambda) -_i c_t \text{ given that there is an acceptance} \quad (34)$$

with φ_i such that $k_{t+1} = \varphi_i k_t$.

Proposition 4 *A solution to the following system*

$$\psi_i^\eta = \frac{\alpha_i}{(1+\psi_i)^{1-\eta}}[\lambda^{1-\eta} + p_i\psi_i(2-\lambda)^{1-\eta} + (1-p_i)\psi_j^\eta\psi_i^{1-\eta}(2-\lambda)^{1-\eta}] \quad (35)$$

with $i, j = 1, 2$ and $i \neq j$ uniquely defined the equilibrium investment path

$${}_iI_t = \frac{\psi_i - (1-\lambda)}{1+\psi_i} k_t \quad (36)$$

if the transversality condition and the immediate-agreement conditions hold. That is, $\alpha_i(p_i\varphi_i + (1-p_i)\varphi_j)^{1-\eta} < 1$ *and* $\mu_j \geq \delta_j\alpha_j\beta_j\varphi_i^{1-\eta}$*, respectively.*

Proof. After some manipulations, the FOC of problem (31)-(34) is as follows.

$$_iI_t = \frac{2^{(1-\eta)/\eta}(\alpha_i\beta_i)^{1/\eta} - (1-\lambda)}{1 + 2^{(1-\eta)/\eta}(\alpha_i\beta_i)^{1/\eta}} k_t \tag{37}$$

This implies that the equation of motion (34) become $k_{t+1} = \varphi_i k_t$ where

$$\varphi_i = \frac{2^{(1-\eta)/\eta}(\alpha_i\beta_i)^{1/\eta}(2-\lambda)}{1 + 2^{(1-\eta)/\eta}(\alpha_i\beta_i)^{1/\eta}} \tag{38}$$

Moreover, from the optimal expected payoff of a responder $\frac{\mu_j k_t^{1-\eta}}{1-\eta}$, by using (37) we obtain the following equation:

$$\mu_j = \frac{1}{2^{(1-\eta)/\eta}} \left[\frac{\lambda}{1 + 2^{(1-\eta)/\eta}(\alpha_i\beta_i)^{1/\eta}}\right]^{(1-\eta)/\eta} + \alpha_j\beta_j\varphi_i^{1-\eta} \tag{39}$$

while for a proposer,

$$\phi_i = \frac{1}{2^{(1-\eta)/\eta}} \left[\frac{\lambda}{1 + 2^{(1-\eta)/\eta}(\alpha_i\beta_i)^{1/\eta}}\right]^{(1-\eta)/\eta} + \alpha_i\beta_i\varphi_i^{1-\eta} \tag{40}$$

Let

$$\psi_i = 2^{(1-\eta)/\eta}(\alpha_i\beta_i)^{1/\eta} \tag{41}$$

then equations (37), (38), (39) and (40) can be re-written as follows.

$$_iI_t = \frac{\psi_i - (1-\lambda)}{1+\psi_i} k_t \tag{42}$$

$$\phi_i = \frac{\lambda^{1-\eta} + \psi_i(2-\lambda)^{1-\eta}}{(1+\psi_i)^{1-\eta}2^{1-\eta}} \tag{43}$$

$$\mu_j = \frac{\lambda^{1-\eta} + \psi_j^{\eta}\psi_i^{1-\eta}(2-\lambda)^{1-\eta}}{(1+\psi_i)^{1-\eta}2^{1-\eta}} \tag{44}$$

$$\varphi_i = \frac{(2-\lambda)\psi_i}{1+\psi_i} \tag{45}$$

In addition, the coefficient β_i is also uniquely defined given (32). In particular, by using (43) and (44),

$$\beta_i = \frac{\lambda^{1-\eta} + p_i\psi_i(2-\lambda)^{1-\eta} + (1-p_i)\psi_j^{\eta}\psi_i^{1-\eta}(2-\lambda)^{1-\eta}}{(1+\psi_i)^{1-\eta}2^{1-\eta}} \tag{46}$$

Then, by using (46), equation (41) can be written as (35). Then a unique solution to (35), uniquely defined the coefficients ϕ_i, β_i, μ_i and therefore the investment level ${}_iI_t$. However, this can only be a solution to the bargaining game if the transversality condition holds, that is, $\alpha_i(p_i\varphi_i + (1-p_i)\varphi_j)^{1-\eta} < 1$ and the immediate agreement condition holds, that is, $\mu_j \geq \delta_j\alpha_j\beta_j\varphi_i^{1-\eta}$. ■

If there is maximum depreciation ($\lambda = 1$), the transversality condition always holds since φ_i is always smaller than 1 (see (42) and therefore, $\alpha_i(p_i\varphi_i + (1-p_i)\varphi_j)^{1-\eta} < 1$). In general, we cannot explicitly solve system (35), since the polynomial is of degree $1/(1-\eta)\eta$. In the remaining of this section, we therefore consider some specific cases in which a solution is possible. First, players are assumed to have the same rate of time preference. Next, we evaluate the equilibrium numerically.

Corollary 5 *When players have the same rate of time preference ($r_i = r$), there is a unique solution to system (35) and the investment level in equilibrium is*

$${}_iI_t = [\alpha(2-\lambda)^{1-\eta}]^{1/\eta}k_t$$

for $\alpha(2-\lambda) < 1$.

Although players can make a proposal with different probabilities, they are symmetric since they have the same rate of time preference and they share the surplus for their own consumption equally. Then, the problem is similar to a standard social planner's problem in which a regulator need to choose how much to invest.

When players have different time preferences, to make the problem more tractable we assume that $\eta = 0.5$ and to simplify the depreciation rate λ is equal to 1. Under these conditions we evaluate numerically the equilibrium and the parametrical effects on the equilibrium outcome.

Result 1 *There is always an equilibrium as long as* α_i *is not too large while* α_j *has an intermediate value for any* $i, j = 1, 2$ *and* $i \neq j$.

To show this we assume that players have the same probability to propose ($p = 1/2$). Then, fig. 1 shows that if α_j is close to 0.999 while α_i is smaller than 0.7, then the equilibrium does not exists (this area is indicated by green dashes, while the black bars indicate that the conditions for the existence on an MPE without delay are satisfied). A similar result can be obtained when the probability to propose varies (see result 3 below). This result implies that in the most likely case in which parties have discount factors very close to 1, an equilibrium without delays exists.

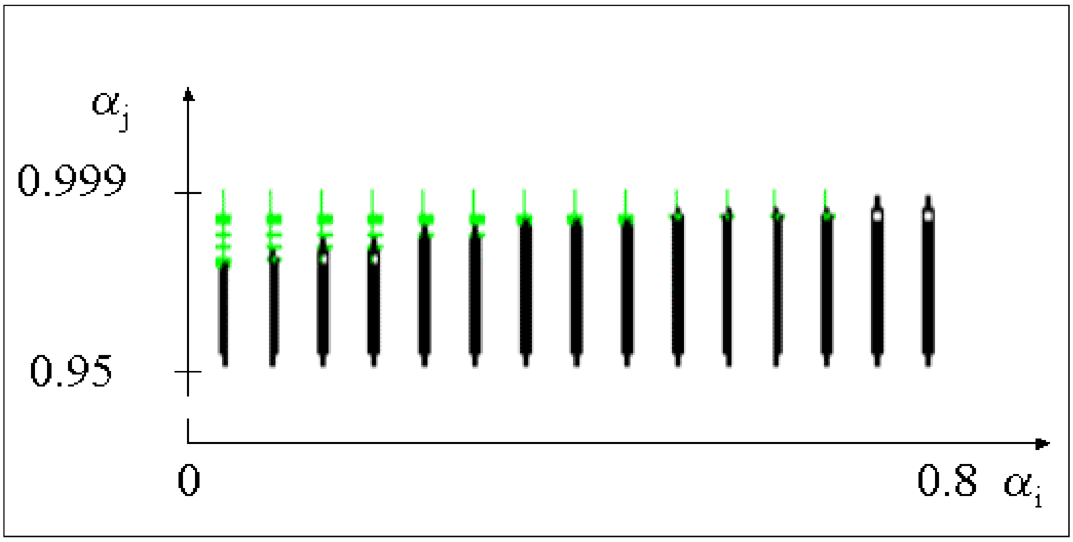

Fig. 1 Area with no MPE is indicated by green crosses for p=1/2.

Result 2 *The most patient player invests more than his opponent for a given level of capital stock* k_t*.*

Fig. 2 below shows the case in which parties can propose with equal probability, however, this result can be shown for any value of p_i. The red circles indicate the area in which player i invests more than player j, vice-versa for the black diamonds. The green crosses indicate the case in which there is no MPE (that is for α_1, α_2 exactly equal to either 0 or 1 in this less detailed graph. See fig. 2 for α_i in (0,1)). This result is intuitive: if a player is relatively more patient than his opponent, then he prefers a higher investment plan.

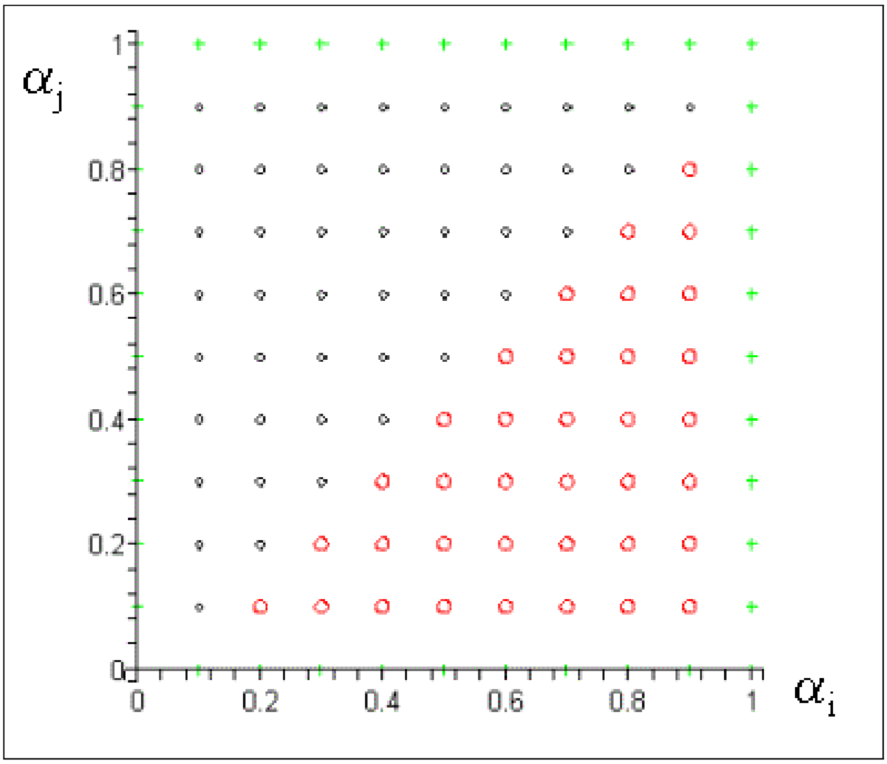

Fig. 2 Player i invests more than player j whenever $\alpha_i > \alpha_j$ for any i and j.

Result 3 *When a player can propose more often then the other player then the area in which an MPE does not exist is slightly modified.*

This result can be shown by comparing fig. 1 ($p_i = 0.5$) with fig. 3 ($p_i = 1/3$). In the latter the green area is slightly smaller for $\alpha_i < 0.5$ and just the same for larger α_i. However, when the focus is on α_i very large and α_j at intermediate levels, the green crossed area is slightly larger. In other words, the area in which there is no MPE becomes asymmetric when p_i differs from 1/2.

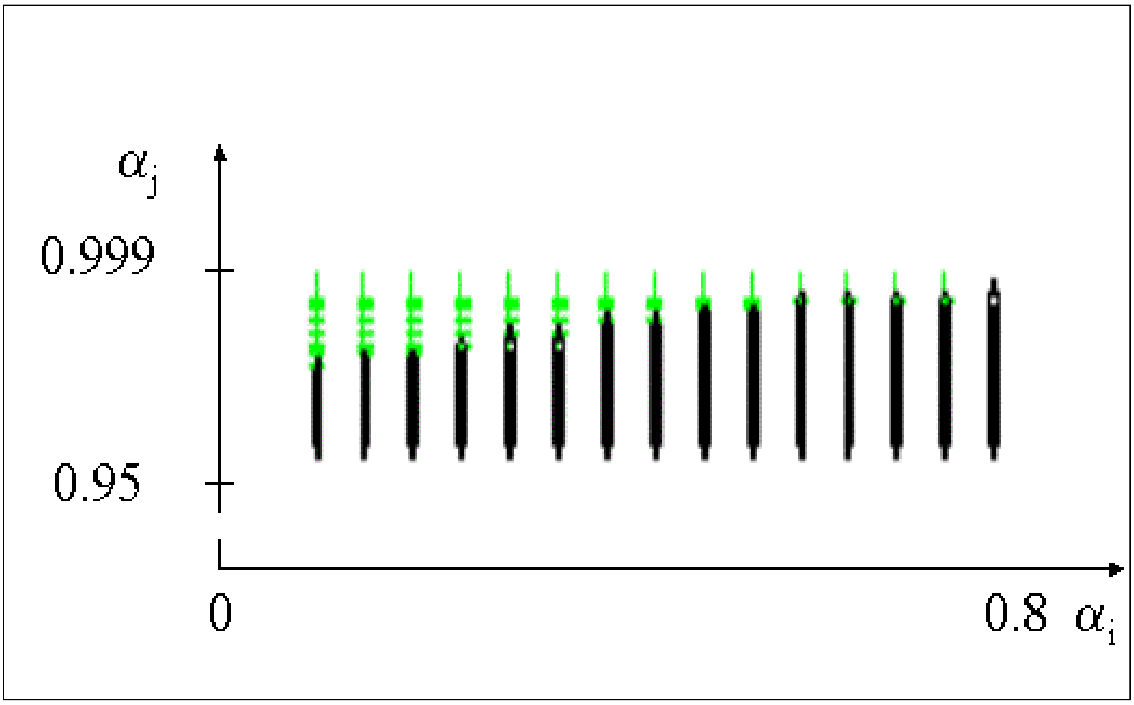

Fig. 3 Area with no MPE is indicated by green crosses for p=1/3.

Result 4 *If player i is more impatient than player j, but the probability that player i proposes increases then player j's level of investment decreases while player i's increases for a given k_t.*

Fig. 4 shows this result for the case in which players' between-cake discount factors are as follows $\alpha_i = 0.7$ and $\alpha_i = 0.8$. This result can be shown for any general asymmetry in players' rate of time preference, as long as there is an MPE. Since the investment level ${}_iI_t$ is an increasing function of ψ_i, then from in fig. 4, we can conclude that although player i always invests less than player j for a given level of capital stock k_t, player $i's$ level of investment, for a given k_t, increases with p_i, while player $j's$ decreases with p_i.

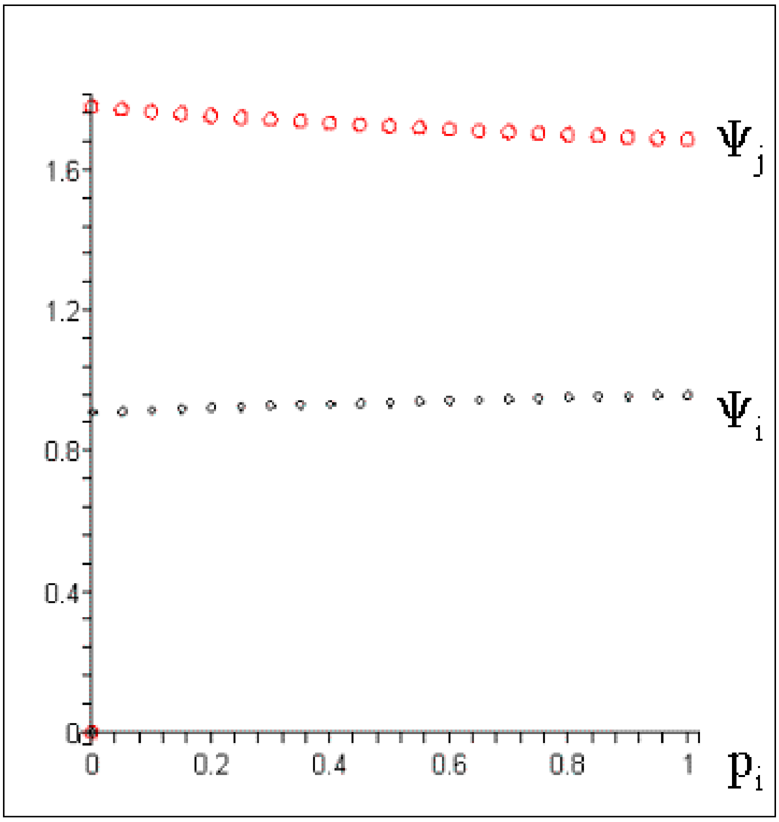

Fig. 4 Auxiliary valiables for p_i in (0,1) when $\alpha_i = 0.7$ and $\alpha_i = 0.8$.

The intuition is that a more patient player always prefers to invest more than his opponent. However, if his opponent is more likely to make an offer, then it is more likely that the total share of the surplus consumed increases (since the less impatient player will reduce the level of investment, for a given capital stock, as soon as he can make an offer), then the high investment made by a patient player is partially lost.

For this reason, when the probability of proposing increases for the impatient player, the patient player reduces his investment plan. On the other hand, the other player has to increase the investment level to compensate for the fact that the investment made by his opponent is not as high as in the past and moreover, the opponent is less likely to make an offer. The impatient player also needs to take into account that his offer has to be acceptable to the patient player and therefore he has to increase the level of investment.

5 Final Remarks

A bargaining model with investment decisions can be very complex, since it has a recursive structure (the bellman equation), which embodies another recursive structure (via the immediate agreement condition). However, we can conclude that in an MPE without delay a proposer is able to extract all the surplus not invested if the constraint (the immediate agreement condition) is not binding, otherwise he has to leave a positive share to his opponent.

Moreover, when the parties agree to share the surplus in equal parts before engaging in a bargaining game on the investment levels, we show that there is an equilibrium whenever players' rates of time preference are not too different when one party becomes very patient. Finally, when an equilibrium exists, the most patient player invests more than his opponent, for a given level of capital stock, but he decreases his investment plan if his opponent is more likely to propose.

We conclude this analysis with a comment on the possibility of other MPE. In particular, can an MPE with delays exist? Given the non-stationary structure of the game (in the sense that at similar nodes, for instance whenever player i proposes, the subsequent subgames do not look the same) equilibria with delays cannot be excluded. Since we need an analytical solution of the model to investigate this possibility, here we attempt to give some intuition as to why delays are possible in equilibrium. Suppose that at time t, a proposer (say, i) prefers to make an offer which induces a rejection rather than an acceptance, moreover, this is the only point in time in which a delay is profitable. If at some point a delay is profitable, it must be that what a proposer, i, gets when there are no delays, $v_i(k_t)$, is smaller than what he would get, in expectation, in the continuation game. This implies, $v_i(k_t) < z_i w_i(k_t)$ with $z_i = \delta_i(1 - p_i)/(1 - \delta_i p_i) < 1$, where $w_i(k_t)$ is the expected discounted utilities of a responder. That is, the cost of a rejection has to be small (δ_i large) and the probability to become again a proposer, p_i, has to be small. Moreover, player i's expected utility as a responder must be sufficiently higher than the expected utility of being a proposer. This can hold when a responder is able to invest more than the other player, for a given level of capital stock. The discounting plays an important role. On the one hand, a player with a high discount factor is able to make a large investment. On the other hand, a player who induces a rejection must be sufficiently patient. In other words, players' differences in discounting are marginal. This implies that the cost of a rejection for a proposer is low, but so is the gain. In conclusion, if the gain from a rejection is smaller than the cost of a rejection then a proposer will never have an incentive to make an unacceptable proposal. However, when the gain from a rejection is larger than the costs, then an equilibrium with delays can exist.

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