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Optimal Martingales and American Option Pricing

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Abstract

Pricing American options is an interesting research topic since there is no analytical solution to value these derivatives. Different numerical methods have been proposed in the literature with some, if not all, either limited to a specific payoff or not applicable to multidimensional cases. Applications of Monte Carlo methods to price American options is a relatively new area that started with Longstaff and Schwartz (2001). Since then, few variations of that methodology have been proposed. The general conclusion is that Monte Carlo estimators tend to underestimate the true option price. The present paper follows Glasserman and Yu (2004b) and proposes a novel Monte Carlo approach, based on designing "optimal martingales" to determine stopping times. We show that our martingale approach can also be used to compute the dual as described in Rogers (2002).

Key Words: American options, Monte Carlo method

JEL Classification: G10, G14

I Introduction

Pricing American options is an interesting research area because there is no closed form solution to price these derivatives. Different numerical methods have been proposed in the literature with all the methodologies being either limited to specific payoff functions or not applicable to multidimensional cases. Monte Carlo methods to price American options is a relatively new area that started with Longstaff and Schwartz (LS) (2001)[1]. Few variations of the LS methodology have been proposed in the literature (see for example Rogers, 2002; Glasserman and Yu, 2004a; Cerrato, 2008 amongst the others). The general conclusion is that Monte Carlo estimators underestimate the true option price because they use the least squares rule to determine the optimal stopping times. In fact, since the least squares rule is not an optimal stopping rule, the probability of choosing sub-optimal exercises decisions increases and, consequently, so does the option price bias.

Glasserman and Yu (2004b) implement the LS estimator using martingales basis in the regression and show that the estimator converges, almost surely, to the correct option price. The main problem with their methodology is that the assumption of finite variance imposed on the basis functions might be too restrictive if the basis considered are martingales. Glasserman and Yu (2004a) under the same assumption of martingales basis and Geometric Brownian motion were able to derive the rate of convergence for the typical Longstaff and Schwartz (2001) estimator without imposing a subsequent limit on the number of stochastic paths used in the simulation but rather using joint limit.

Rogers (2002) formulates the American option pricing problem as the dual and shows that one can use a martingale approach to reduce the probability of choosing sub-optimal policies when determining the early exercise value. For general martingales the option price given by the dual will form an upper bound for the true option price. However, if the martingale used is an optimal martingale the option price can be estimated exactly. The main problem with this methodology is that it is unclear how an optimal martingale can be designed. Under certain assump-

[1] In effect Carriere (1996) was the first to propose this approach. It was then extended in Longstaff and Schwartz (2001).

tions the Glasserman and Yu (2004b) and Rogers (2002) methods are similar. Chen and Glasserman (2006) made recently an important theoretical contribution. They suggest an improved additive dual obtained by iterations.

Designing optimal martingales to price American options has interesting empirical applications. However as Rogers (2002) points out this important issue has become more "an art than a science", in the sense that studies in the literature have relied on ad-hock martingales. One of the contributions of this paper is to propose a novel "optimal martingale" approach to determine stopping times. We show that our approach produces very accurate prices. Furthermore the present paper proposes two novel algorithms (lower and upper bounds) based on our optimal martingale. The paper also discusses ways of implementing the LS approach using Black and Scholes prices as basis functions. Finally the methodology is used to price long-dated American options and the accuracy of the options prices estimates investigated.

To quickly compare the estimator in Chen and Glasserman (2006) with the one proposed in the present paper in Sections 3-5, we consider as an example, one of the most difficult options to price as in Chen and Glasserman (2006), (see page 23 in Chen and Glasserman). The put option is deep in the money. The initial stock price $S = \$50$, the strike $K = 100$, the interest rate is 20%, the stock price volatility 30%, and time to maturity one year. The best policy for the option holder should be "exercise immediately". The true option price is \$50. We use the first three martingale basis, 100,000 replications and 70 time steps. After averaging fifty trials the option value is \$49.966, and the standard error 0.00151.

II A General Framework for American Option Pricing

Consider the following probability space (Ω, F, P) and the filtration $(F_i)_{i=0,\ldots,n}$, with n being an integer. Define by $X_0, X_1, ..., X_n$ an R^d valued Markov chain representing a state variable recording all the relevant information on the price of an asset. Assume that $V_i(x)$, $x \in R^d$ is the value of an option exercised at time t_i under the state

x, and $\Theta(X)$ is the options payoff. Following Glasserman and Yu (2004a) the option pricing problem is defined by the following dynamic programming framework[2]:

$$V_i(x) = \sup_{\tau \in \Gamma} E[\Theta_\tau(X_\tau)|X_i = x] \tag{1}$$

$$V_n(x) = \Theta(x) \tag{2}$$

The value of an American option at time $t_{i,}$ under the state $X_i = x$, is given by maximizing its expected payoff over all possible stopping times $\tau \in \Gamma$ (equation 1) with final condition given by equation (2). Combining equations (1)-(2) we have that the value of an American option at time t_i is given by the maximum between its value if immediately exercised and its expected value (i.e. the continuation value).

$$V_i(x) = \max\{\Theta_i(x), E[V_{i+1}(X_{i+1})|X_i = x]\} \tag{3}$$

Finding the continuation value in (3) is a difficult task since it involves solving an optimal stopping problem. Different solutions have been proposed in the literature, as for example replacing it with the simple regression

$$E[V_{i+1}(X_{i+1})|X_i = x] = \sum_{k=0}^{K} \beta_{ik}\psi(x) \tag{4}$$

Equation (4) can also be written in terms of the option‘s continuation value C_i

$$C_i(x) = E[V_{i+1}(X_{i+1}|X_i = x] \tag{5}$$

where C_i is a linear combination of the coefficients in (4)

$$C_i(X_i) = \beta^{'}\psi_i(X_i) \tag{6}$$

[2]For simplicity we do not consider discounted payoffs.

with $\beta_i^{'} = (\beta_{i1}, \beta_{i2}, ...\beta_{iK})^{'}$being the regression coefficients and $\psi_i(X_i) = [\psi_1(X_1), \psi_2(X_2), ..., \psi_K($ some specific basis functions.

Computing the option price from (1)-(3) is rather demanding and therefore one has to rely on approximations. The typical assumption made is that $V_i(.)$ is a function spanning the Hilbert space, therefore the conditional expectation can be approximated by the orthogonal projection on the space generated by a finite number of basis functions ψ_{ik} , $i = 1, 2, .., n$ and $k = 0, 1, ..., K$. If we replace (1)-(3) with their sample quantities, we have

$$V_n^*(x) = \Theta_n(x) \tag{7}$$

$$V_i^*(x) = \max\{\Theta_i(x), E[V_{i+1}(X_{i+1})\,|X_i = x|]\} \tag{8}$$

One can now use a simple regression to estimate the conditional expectation in (8)

$$E[(V_{i+1}(X_{i+1})|X_i) = \sum_{k=0}^{K} \beta_{ik}\psi_{ik}(X) + \varepsilon_{i+1} \tag{9}$$

Equation (9) will hold exactly since we have introduced the residual ε_{i+1} on its right hand-side. The advantage of working with equation (9) is that, as we shall see, its coefficients can be easily computed by Least Squares. Lemma 1 defines the asymptotic convergence of the Least Squares estimator.

Lemma 1 *if $E(\varepsilon_{i+1}|X_i) = 0$ and $E[\psi_i(X_i)\psi_i(X_i)^{'}]$ is finite and non-singular then $V_i^* \rightarrow V_i$ for all i*

See Longstaff and Schwartz (2001), Clement et al (2002), Glasserman and Yu (2004a) for a proof.

Various proofs of convergence of this estimator have been discussed in Longstaff and Schwartz (2001), Clement et al (2002), Glasserman and Yu (2004a). In Equation

(9) the conditional expectation has been estimated using current basis functions (i.e. ψ_{ik}, $i = 1, 2, ..., n$). As explained in Glasserman and Yu (2004b) the option price at time $i + 1$ is likely to be more closely correlated with the basis function $\psi_{i+1}(X_{i+1})$ rather than $\psi_{ik}(X_i)$. Glasserman and Yu (2004b) suggest a methodology based on Monte Carlo simulations where the conditional expectation is approximated by $\psi_{i+1}(X_{i+1})$. They show that their methodology has a regression representation given by

$$\widehat{V}_{i+1}(X_{i+1}) = \sum_{k=0}^{K} \varpi_{ik} \psi_{i+1,k}(X_{i+1}) + \widehat{\varepsilon_{i+1}} \tag{10}$$

where ϖ_{ik} are k coefficients generally estimated by least squares. Proof of convergence in this case requires using martingales basis functions

Definition 1 *(Martingale property of basis functions)* $E(\psi_{i+1}(X_{i+1})|(X_i) = \psi_i(X_i)$ *for all* i

Under Definition 1, Glasserman and Yu show that regressions (9) and (10) are equivalent but standard errors from regression later are smaller. Option prices in (9) and (10) are linear combination of the same basis functions. They only differ in the way coefficients are estimated. Glasserman and Yu (2004b) call this method regression later, since it involves using $\psi_{i+1}(X_{i+1})$. On the other hand, they call the LS (2001) method regression now since it uses the basis $\psi_i(X_i)$. Note that as a consequence of Definition 1 we now have that $E(\widehat{\varepsilon}|X_i) = 0$ and therefore the conditional expectation is approximated exactly. However, in this case, the finite variance assumption imposed on the basis functions might become restrictive. We believe our martingale approach should make Assumption 2 in Glasserman and Yu (2004b) more likely to hold.

An alternative way to formulate the option pricing problem has been suggested by Rogers (2002). Rogers (2002) shows that the option pricing problem can be formulated in terms of minimizing a penalty function given by a class of martingales over the lifetime of the option. While any martingales will produce an upper bound around the true option price, an optimal martingale will estimate it exactly.

To see this, note that a consequence of the dynamic programming framework in (1)-(3) is that the option price is a supermartingale. Therefore one can use the Doob-Meyer decomposition for martingales and write

$$V_i = V_0 + M_i - A_i \tag{11}$$

where M_i is a martingale with $M_0 = 0$ and A_i a previsible non-decreasing process with $A_0 = 0$.

Rogers (2002) shows that under the dual the value of the option at t_0 is given by

$$V_0 = \inf_{M \in H_0} E[\sup(V_i - M_i)] \tag{12}$$

where H_0 is the space of all martingales and the infinitum is obtained when an optimal martingale $M = M^*$ is chosen.

Therefore under this martingale one can price options exactly. However, this result holds if the martingale chosen is an optimal martingale. As noted in Rogers (2002) determining an optimal martingale turns out to be at least as difficult as solving the original option pricing problem! One of the contributions of the present paper is to build on Glasserman and Yu (2004b) and Rogers (2002) and propose a simple way to design optimal martingales.

III A Simple Approach to Designing Optimal Martingales

Before introducing our approach, let us first clarify what we mean by "optimal martingale". Suppose that Definition 1 holds and define the following random variable M^*

$$M_i^* = V_{i+1}(X_{i+1}) - E[V_i(X_i)|X_{i-1}] \tag{13}$$

with $M_0 = 0$

Lemma 2 *Let μ^2 be the space of martingales M^* bounded in L^2 such that any M_i^* in μ^2 is an R martingale and therefore* $\sup E(M_i^{*2}) < \infty$ *. The space μ^2 inherit the Hilbert structure from $L^2(\infty)$*

See Appendix 1 for a proof.

Re-write (13) using estimated quantities as

$$M_i^* = V_{i+1}^*(X_{i+1}) - C_i^*(X_i) \tag{14}$$

$$= V_{i+1}^*(X_{i+1}) - \sum_{k=0}^{k} \beta_{ik}^*[\psi_i(X_i)] \tag{15}$$

Under Definition 1, we have

$$M_i^* = C_i^*(X_i) - \beta^{*'}\psi_i(X_i) \tag{16}$$

and therefore it follows that if $M_0^* = 0$, the process is a martingale

$$E[(M_{i+1}^*)|X_1, X_2, ...X_i, ..., \beta] = M_i^* \tag{17}$$

An immediate consequence of Lemma 2 is that the martingale M^* belongs to a specific class of martingales. This martingale is well defined and different than others proposed in the literature (see for example Rogers).

The next section clarifies the link between the martingale approach introduced in this section and American options pricing.

IV Optimal Martingales and American Option Pricing

Using the martingale M_i^* one can design basis functions[3] as $M_{ik}^*(X_i)$. Thus equations (9) or (10) become

$$C_i(X_i) = \sum_{k=0}^{K} a_{i,k} M_{ik}^*(X_i)$$

Consider the sample version of this equation

$$C_i^*(X_i) = \sum_{k=0}^{K} a_{ik}^* M_{ik}^*(X_i) \tag{18}$$

where $a_i^* = (a_{i0}^*, a_{i1}^*, .., a_{iK}^*)'$are least squares coefficients

Theorem 1 *If Definition 1 and Lemma 2 hold then $C_i^* \to C_i$ and $V_i^* \to V_i$ for all i*

See Appendix for proof.

Remark 1 *The proposed approach is similar in spirit to the one suggested by Glasserman and Yu (2004b) but the important difference is that we suggest martingales that are bounded in L^2 and a novel algorithm based on this martingale to compute the dual.*

V A Simple Algorithm for American Option Pricing

In this section we show how our approach can be used to extend the LS (2001) method. A simple algorithm, that can be extended to price exotic options, is presented below

[3] Notation 1 in the Apenndix gives further details on this.

1)At each time t_i, in a recursive fashion, use a regression approach that satisfies Definition 1. Start, for example, with two basis functions and save the residuals.

2)Repeat the step in (1) increasing the number of basis from two to three, and save the residuals.

3)Use the residuals in (1) and (2) to obtain L^2martingales as explained in Section III[4]. Run a new regression using the two martingales basis and estimate the conditional expectation.

4)Repeat (1)-(3) in a recursive fashion.

Remark 2 *The proposed approach seems to be computationally inefficient. However note that this multiple regressions approach does not impact massively on the computational speed. The approach in Rogers (2002), for example, would be more inefficient than ours in terms of computational speed. Intuitively, using our approach we would expect each martingales obtained by iterative least squares regressions to have smaller and smaller variance as we increase the number of regressions.*

The empirical results are reported in Table 2 and discussed in the next section. We have used five and six basis functions in the first regression and three and four in the second.[5]

VI Empirical results

To start with, we price an American put options written on a stock. The empirical example follows very close Longstaff and Schwartz (2001) and therefore assumes the same parameters as in that study. Some of the parameters such as stock price, strike, volatility and time to maturity are reported in the tables. s is the initial stock price, k is the strike, v the stock price volatility, and finally T the time to

[4]At this point one can also use cross-products to increase the number of basis. Please refer to Notations 1 in the Appendix for more details.

[5]Note that numbers at the top of the tables refer to the basis functions used in the two regressions.

expiry of the option.[6] The short term interest rate is assumed to be 6%p.a. and we use 100,000 replications, 50 time steps and antithetic variates. Standard errors and root mean squares errors were calculated as in Cerrato (2008) and obtained from fifty trials.

We start with a simple martingale. We use the discounted Black & Scholes price as martingale basis. In fact, this is first martingale basis that one should consider.

	Order	3	4	bin	LS price
s=36	value	4.469	4.479	4.4867	4.479
k=40	SE	[0.009]	[0.0065]		[0.007]
v=0.2	RMSE	[0.0198]	[0.0101]		[0.013]
T = 1					
s=36	value	4.827	4.832	4.8483	4.834
k=40	SE	[0.006]	[0.011]		[0.007]
v=0.2	RMSE	[0.0226]	[0.0195]		[0.015]
T = 2					
s=36	value	7.098	7.116	7.1092	7.103
k=40	SE	[0.0093]	[0.0077]		[0.009]
v=0.4	RMSE	[0.0143]	[0.0100]		[0.015]
T = 1					
s=36	value	8.495	8.501	8.5142	8.488
k=40	SE	[0.0123]	[0.0121]		[0.009]
v=0.4	RMSE	[0.0229]	[0.0180]		[0.014]
T = 2					

Table 1: Black and Scholes Basis

We price long dated options since it is well known that standard methodologies do not perform well in this case (see for example Barone Adesi and Whaley (1987)). On the other hand binomial or finite difference methods are inefficient and not applicable to multidimensional problems. In Table 1, we compare two regression methods. The first uses Black & Scholes prices as martingales (see columns three and four), the second uses simple exponential basis functions (see column six). We also report prices obtained by binomial methods (see column five) with 10,000 time steps. We assume this price to be the true price. Estimated prices using Black and

[6] We have also calculated standard errors (SE) and root mean squares errors (RMSE). These are reported in brakets.

Scholes basis are in general higher than the ones obtained by the LS method. Root mean squares errors are generally smaller when four basis functions are considered.

We now turn to the martingale approach described in Sections III and IV. Table 2 shows the empirical results. We compare our methodology with the recent methodologies proposed in Longstadd and Schwatz (2001) and Glasserman and Yu (2004b). To implement our methodology, we use up to six basis functions in the first regression and three and four martingales basis in the second. Three basis are sufficient to obtain a good fit. Standard errors are small and overall of the same order of magnitude as the root mean squares errors. Generally our methodology produces standard errors and root mean squares errors that are lower than the LS (2001) methodology. Standard erros and root mean squares errors of the Glasserman and Yu (2004b) methodology are intead generally higher. These results may suggest that our simple martingale approach can reduce the probability of choosing a sub-optimal strategy when determining stopping times. Note that to implement our method one has to apply multiple regressions. Furthermore basis functions in the first regression must be martingales. Following Cerrato (2008) we use martingales obtained from exponential functions under Geometric Brownian motion.

$$\phi_{ik}(X_i) = (X_i)^k \exp - (kr + k(k-1)\sigma^{2/2})(t_i - t_0)$$

	Order	5 and 3	5 and 4	6 and 3	GY(2004l	bin	LS price
s=36	value	4.473	4.482	4.4809	4.467	4.4867	4.472
k=40	SE	*[0.007]*	*[0.006]*	*[0.005]*	*[0.005]*		*[0.007]*
v=0.2	RMSE	*[0.0153]*	*[0.0076]*	*[0.0077]*	*[0.020]*		*[0.013]*
T = 1							
s=36	value	4.832	4.838	4.832	4.811	4.8483	4.821
k=40	SE	*[0.003]*	*[0.004]*	*[0.006]*	*[0.008]*		*[0.007]*
v=0.2	RMSE	*[0.0166]*	*[0.0110]*	*[0.0200]*	*[0.014]*		*[0.015]*
T = 2							
s=36	value	7.103	7.117	7.101	7.101	7.1092	7.091
k=40	SE	*[0.012]*	*[0.009]*	*[0.009]*	*[0.010]*		*[0.009]*
v=0.4	RMSE	*[0.0138]*	*[0.0117]*	*[0.0121]*	*[0.016]*		*[0.015]*
T = 1							
s=36	value	8.501	8.518	8.491	8.491	8.5142	8.488
k=40	SE	*[0.01]*	*[0.01]*	*[0.009]*	*[0.009]*		*[0.009]*
v=0.4	RMSE	*[0.0166]*	*[0.0117]*	*[0.0257]*	*[0.029]*		*[0.014]*
T = 2							
s=38	value	3.252	3.253	3.247	3.241	3.2572	3.244
k=40	SE	*[0.008]*	*[0.008]*	*[0.009]*	*[0.005]*		*[0.005]*
v=0.2	RMSE	*[0.0098]*	*[0.0089]*	*[0.0136]*	*[0.0169]*		*[0.006]*
T = 1							
s=38	value	3.735	3.754	3.748	3.721	3.7514	3.3735
k=40	SE	*[0.006]*	*[0.009]*	*[0.009]*	*[0.008]*		*[0.006]*
v=0.2	RMSE	*[0.0174]*	*[0.0094]*	*[0.0097]*	*[-0.027]*		*[-0.014]*
T = 2							
s=38	value	6.147	6.156	6.148	6.137	6.1545	6.139
k=40	SE	*[0.007]*	*[0.001]*	*[0.009]*	*[0.0093]*		*[0.006]*
v=0.4	RMSE	*[0.0105]*	*[0.0102]*	*[0.0112]*	*[-0.018]*		*[-0.005]*
T = 1							
s=38	value	7.666	7.68	7.66	7.651	7.6749	7.669
k=40	SE	*[0.01]*	*[0.008]*	*[0.009]*	*0.0121]*		*[0.0098]*
v=0.4	RMSE	*[0.0134]*	*[0.0095]*	*[0.0174]*	*[-0.0291]*		*[-0.0122]*
T = 2							

Table 2: Martingales Basis Functions

Order refers to the polynomial order. S is the stock price, k the option strike, v the stock price volatility and T the time to maturity. SE and RMSE are standard errors and root mean squares srrors. GY(2004b) is the Glasserman and Yu methodology.

VII Computing Upper Bounds

Following Rogers (2002) researchers have proposed different methodologies to compute the dual. Glasserman and Yu (2004b), for example, show that, if Definition 1 holds, one can use simulations to obtain upper and lower bounds with a minimal effort. The martingale approach described in the previous sections can also be extended to this case with little extra effort. We first describe our algorithm, subsequently we discuss the main differences with the existing methods. Our approach follows Rogers and it is based on an additive dual as opposed to the multiplicative dual suggested in Jamshidian (2003). We use this approach since, as discussed in Chen and Glasserman (2006), it produces estimators of the upper bound with the lowest variance. Our goal is to obtain a fast algorithm to compute upper bounds and see if there is scope for further reduction in the variance. Following Rogers (2002) the dual is given by the right hand side of equation (19) below

$$E[\Theta_\tau(X_\tau) \leq V_0(X_0) \leq E[\max_{i=0,1\ldots}(\Theta_n(X_n) - M_n^*)] \tag{19}$$

As discussed in Rogers (2002), any martingales will generate an upper bound in (19). However, equation (19) will hold with equality only if the martingale used is an optimal martingale. Our approach is simpler and more efficient than the ones discussed in the literature (see for example, Haugh and Kogan, 2004; Rogers, 2002; and Glasserman and Yu, 2004b). It can be summarized as follows

1) Use the algorithm described in the previous section to compute at each $t_i$estimates of the martingales basis.

2) Use the out-put in step (1) to obtain an estimate of the conditional expectation.

3)Along each path,compute the summation $M_n^* = \sum_{i=0}^{n-1}[V_{i+1}^*(X_{i+1}) - C_i^*(X_i)]$.

4)Along each path, compute the $\Theta_n^* = \max(\Theta_n, V_n(X_n))$.

5)Estimate the right hand side of Equation (19) along each path.

6)Repeat 1-5 and iterate across each simulated path to compute the option price[7]

[7]As Glasserman (2004) pointed out, martingales basis in this case can be obtained in a trivial

Remark 3 *The dual estimator proposed in this study is similar in spirit to the one suggested in Glasserman and Yu (2004b) but the martingale $M_n^*$is obtained directly from residuals of regression (10). Furthermore, it has also similarities with the approach proposed in Chen and Glasserman (2006)*[8]*. In effect, the rationale for the iterative regression scheme above is that by designing martingales from ordinary least squares residuals one would expect to obtain martingales with smaller and smaller variance as we increase the number of basis functions. Therefore the iterative scheme is similar in spirit to the one proposed in Chen and Glasserman (2006). If the martingale used is an optimal martingale,we would expect the difference between upper (Table 3) and lower (Table 2) bounds to be very small.*

Proof of convergence of this estimator can be obtained using Theorem 1 along the lines as in Rogers (2002). It is worth stressing again that this estimator of the upper bound can be obtained at a minimal effort and therefore it has noticeable advantages with respect to the upper bound estimators suggested in the literature.

VIII Empirical Results

Table 3 shows the empirical results for the upper bounds using the dual algorithm proposed in this study. For completeness, as we have done before, we also report prices estimates using the Longstaff and Schwartz (2001), and Glasserman and Yu

way. In fact in Section 3 we suggest using the regression residuals from regression later and the initial ad-hock condition.

[8]That is, they both rely on iterations. Note that although in Lemma 5.4 Chen and Glasserman (2006) point out that their scheme converges even for processes that are not necessarerly martingales, on the other hand, they had to characterise the process as a martingale to prove Proposition 6.6. Therefore, yet the problem of finding a valid martingale remains an open issue.

(2004b) methods.

	Order	5 and 3	bin	GY(2004l	LS price
s=36	value	4.488	4.4867	4.467	4.472
k=40	SE	[0.006]		[0.005]	[0.007]
v=0.02	RMSE	[0.0013]		[0.020]	[0.013]
T = 1					
s=36	value	4.849	4.8483	4.811	4.821
k=40	SE	[0.007]		[0.008]	[0.007]
v=0.02	RMSE	[0.0004]		[0.014]	[0.015]
T = 2					
s=36	value	7.11	7.1092	7.101	7.091
k=40	SE	[0.011]		[0.010]	[0.009]
v=0.04	RMSE	[0.0001]		[0.016]	[0.015]
T = 1					
s=36	value	8.519	8.5142	8.491	8.488
k=40	SE	[0.008]		[0.009]	[0.009]
v=0.04	RMSE	[0.0057]		[0.029]	[0.014]
T = 2					
s=38	value	3.258	3.2572	3.241	3.244
k=40	SE	[0.009]		[0.005]	[0.005]
v=0.02	RMSE	[0.0001]		[0.0169]	[0.006]
T = 1					
s=38	value	3.752	3.7514	3.721	3.3735
k=40	SE	[0.009]		[0.008]	[0.006]
v=0.02	RMSE	[0.0001]		[-0.027]	[-0.014]
T = 2					
s=38	value	6.158	6.1545	6.137	6.139
k=40	SE	[0.011]		[0.0093]	[0.006]
v=0.04	RMSE	[0.0035]		[-0.018]	[-0.005]
T = 1					
s=38	value	7.681	7.6749	7.651	7.669
k=40	SE	[0.013]		0.0121]	[0.0098]
v=0.04	RMSE	[0.0057]		[-0.0291]	[-0.0122]
T = 2					

Table 3: Dual Mathod and Upper Bounds

Order refers to the polynomial order. S is the stock price, k the option strike, v the stock price volatility and T the time to maturity. SE and RMSE are standard errors and root mean squares srrors. GY(2004b) is the Glasserman and Yu methodology.

Table 3 reports results with five martingales basis in the first regression and 3 in the second regression. We consider long dated options, that, in general, are more difficult to price and compute standard errors and absolute errors. We use 100,000 replications and fifty time steps. Prices are averages of fifty trials. The upper bounds are very close to the true price. The other two methodologies do not reach the same accuracy (in terms of standard errors and root mean squares errors). Absolute errors are smaller for all the combinations of input parameters. The low standard errors show that the sample uncertainty is relatively modest. The computation of the upper bound takes about 7 seconds on a standard Intel Core 2 processor. Given the efficiency gain and the options prices estimates in Table 3, we believe that our methodology is also very relevant for practitioners.

IX Conclusions

Monte Carlo method to price American options is now an active research area. In fact this methodology can be easily extended to account for path dependency or multi-dimensionality. Longstaff and Schwartz (2001) suggested using least squares approximation to approximate the option price on the continuation region and Monte Carlo methods to compute the option value. Proofs of asymptotic convergence of the LS estimator are derived under various assumptions and therefore more work is needed in this case. Clement et al (2002) showed that the LS option price converges, almost surely, to its true price. But the theoretical proof in Clement et al (2002) has some limitations in that it is based on a sequential rather than joint limit.

Glasserman et al (2004a) considered the limitations in Clement et al (2002) and proved convergence of the LS estimator as the number of paths and the number of polynomials functions increase together. However, the assumption of martingales polynomials is required in this case. Glasserman and Yu (2004b) implemented the LS estimator using martingales basis in the regression and showed that the estimator converges to the correct option price.

In this study we proposed a novel approach to designing optimal martingales to price American options. We proposed two novel algorithms (upper and lower bound) based on our optimal martingale and showed that the estimated options prices are precise and not disperse. The methodologies are simple to implement and computationally efficient and therefore very relevant for practitioners. Extensions of our methodologies to price path dependent options and basket options are left on the agenda for future research.

X Appendix

Proof of Lemma 2

Lemma 2 specifies the type of martingale considered and shows that the space spanned by this martingale inherits a Hilber structure.

First note that if $M^*$is in L^2then $\sup_{i>0} EM^{*2} < \infty$.Define $||M^*||^2 = E(M^{*2}_\infty)$ the norm for the μ^2martingales M^*.Jensen inequality implies

$$E(M^{*2}_\infty|F_i) \geq E(M^*_\infty|F_i)^2 \tag{20}$$

$$E(M^*_i)^2 \leq E(M^{*2}_\infty|F_i) \tag{21}$$

Then it follows that

$$M^*_i : \mu^2 \longrightarrow L^2(F_\infty) \tag{22}$$

Proof of Theorem 1

First note that if the process $M_1, M_2, ...$is a martingale then it follows that $C^*_i$is a martingale and therefore $\sup EC^{*2}_i = ||C^{*2}_\infty||^2_2 = E[C^{*2}_\infty] < \infty$.

Definition 2 *Let $C^*(C^*_i)$ be a martingale such that $E[C^{*2}_i] < \infty$. It follows, using Lemma 2, that $C^* \in \mu^2 \rightarrow L^2(\infty)$.Also, let $C_i := E(C_\infty|F_i)$, where C_∞ is the the limit of the sequence C^*_i*

The Doob's L^2inequality implies that

$$\sup_{i>0} |C_i^* - C_i| \longrightarrow 0 \tag{23}$$

is in L^2

Define the value function

$$V_i^*(X_i) = \max(\Theta_i, C_i^*(X_i)) \tag{24}$$

with X_0 fixed, we have

$$C_0^*(X_0) = \frac{1}{R}\sum_{j=1}^{R} V_1^*(X_{i,j}) \tag{25}$$

$$V_0^*(X_0) = \max(\Theta_0(X_0), C_0^*(X_0)) \tag{26}$$

Since $C_0^* \to C_0$, it follows that $V_0^* \to V_0$

Notation 1 *Martingales Basis Functions*

We now clarify how the regression in Section IV has been implemented using the martingale M_i^*. As mentioned the approach we suggest is a multiple regression approach. In the first regression, we have used regression later and the martingale suggested in Cerrato (2008) as well as in this paper on page 13. For example, suppose we are considering three martingales basis. In this case we can start with, say, two regressions, using regression laters, and increasing in each the number of martingales basis (i.e. first regression one martingale basis, second regression two martingale basis). We then save the residulas from each of these two regressions. The martingale M_i^* can now be computed using the residuals and the approach described in Section III. Using this martingale one can now specify the basis functions for the second regression. We have specified three and four martingales basis (in *the second regression) to compute the prices in Table 2. For example with three basis functions we have considered $\{c, S_t e^{-rt}, M^*_{1t} M^*_{2t},\}$, where $M^*_{j,t}, j = 1, 2$ are the martingales obtained from the first two regressions. Cross section products of these quantities can also be used to increase the number of basis.*

Notation 2 *The Dual algorithm works in the same way. However, in order to use equation (19) we now need to compute the discounted value of the quantity in equation (1).This can be easily obtained from the estimation of the conditional expectation and using the martingale approach suggested in this paper. Furthermore we also need to compute the martingale M^*_n. The latter is not difficult to obtain given that it is given by summing up M^*_i, from $i, 1, ..., n$.*

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