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On the Adaptation of the Lagrange Formalism to Continuous Time Stochastic Optimal Control: A Lagrange-Chow Redux

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Abstract

We show how the classical Lagrangian approach to solving constrained optimization problems from standard calculus can be extended to solve continuous time stochastic optimal control problems. Connections to mainstream approaches such as the Hamilton-Jacobi-Bellman equation and the stochastic maximum principle are drawn. Our approach is linked to the stochastic maximum principle, but more direct and tied to the classical Lagrangian principle, avoiding the use of backward stochastic differential equations in its formulation. Using infinite dimensional functional analysis, we formalize and extend the approach first outlined in Chow (1992) within a rigorous mathematical setting using infinite dimensional functional analysis. We provide examples that demonstrate the usefulness and effectiveness of our approach in practice. Further, we demonstrate the potential for numerical applications facilitating some of our key equations in combination with Monte Carlo backward simulation and linear regression, therefore illustrating a completely different and new avenue for the numerical application of Chow's methods.

Keywords: Lagrange formalism, continuous optimization, dynamic programming, economic growth models, stochastic processes, optimal control, regression-based Monte Carlo methods

JEL: C61, C63, C65, E22

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1 Introduction

Many problems in Economics are of the following type: an objective functional measuring economic performance needs to be optimized subject to a constraint accounting for the scarcity of resources. The classical Lagrange multiplier theorem, which is taught to virtually all undergraduate economics students, provides the right tool to solve such problems mechanically, at least if the problem is static and does not involve random shocks. A correct formulation of the Lagrange multiplier theorem however is necessary, to apply it correctly and prevent making conclusions that the theorem does not allow per se.[1] Lang (1973), which for decades has served as one of the main undergraduate texts in multi-dimensional calculus states the classical Lagrange multiplier theorem as follows:

Theorem 1. *Let $U \subset \mathbb{R}^n$ be an open set and $g : U \to \mathbb{R}$ be a continuously differentiable function and let S be the set of points $x \in U$, s.t. $g(x) = 0$ but*

$$\nabla g(x) \neq 0. \tag{1}$$

Further, let $f : U \to \mathbb{R}$ be continuously differentiable and assume that x^* is a maximum for f on S, i.e. $f(x^*) \geq f(x)$ for all $x \in S$. Then there exists a real number λ such that[2]

$$\nabla f(x^*) = \lambda \cdot \nabla g(x^*). \tag{2}$$

The theorem provides a necessary but in general not a sufficient condition for an extremum under a single constraint. Under appropriate convexity conditions, a sufficient criterion can be derived. For the case of multiple constraints, the function g can be chosen vector-valued, i.e. $g : U \to \mathbb{R}^m$, in this case, the Lagrange multiplier will be a vector in $\mathbb{R}^m$ and equation (2) will become

$$Df(x^*) = \lambda^\top \cdot Dg(x^*), \tag{3}$$

where $Df(x^*)$ and $Dg(x^*)$ denote the corresponding Jacobian matrices, i.e. $Df(x^*) = \nabla f(x^*)^\top$ and $Dg(x^*) = \left(\frac{\partial g_i}{\partial x_j}\right) \in \mathbb{R}^{m\times n}$. Note that the product on the right-hand side is a product between an $\mathbb{R}^{1\times m}$ vector and an $\mathbb{R}^{m\times n}$ times matrix resulting in a $\mathbb{R}^{1\times n}$ vector. The fact that the right-hand side of the equation involves the transpose of the vector λ indicates that λ should actually be considered to be in the dual space of $\mathbb{R}^m$, i.e. as a linear map $\lambda : \mathbb{R}^m \to \mathbb{R}$, with $Dg(x^*) : \mathbb{R}^n \to \mathbb{R}^m$ and $Df(x^*) : \mathbb{R}^n \to \mathbb{R}$ linear maps of the corresponding spaces. Note that differentials are defined as linear maps, rather than as numbers, euclidean vectors or matrices, and the product on the right-hand side of equation (3) corresponds to the composition of two linear maps.

Finite dimensional vector-valued constraints can be used to implement dynamic constraints in

[1] Unfortunately treatment of the Lagrange multiplier formalism in many undergraduate maths texts for economists is light touch, focusing on intuition and examples, and often conveys the perception that it is virtually always applicable as long as the Lagrange function can be written down.

[2] Here $\nabla f(x^*)$ denotes the gradient of the function $f(\cdot)$ at the point x^* i.e. $\nabla f(x^*) = \left(\frac{\partial f}{\partial x_1}(x^*), ..., \frac{\partial f}{\partial x_n}(x^*)\right)^\top$ and similar for $\nabla g(x^*)$.

discrete time with a finite horizon as well as finitely distributed random shocks. This requires more sophistication but is standard in dynamic Macroeconomic modeling. For continuous time or the inclusion of random shocks with a continuous distribution, however, the Lagrange multiplier theorem does not apply a priori. In this case, the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ above need to be replaced by infinite dimensional spaces and it is not a priori clear how to take differentials on these spaces. Clarke (1976) provided a Lagrange-type theorem that holds on Banach spaces and our work can be seen as a continuation of this work, systematically adapted to cover continuous time stochastic optimal control problems. In fact, if the relevant spaces admit a Fréchet structure, then using Fréchet derivatives in place of the classical finite-dimensional derivatives permits a Lagrange multiplier theorem, which as we show can be used to derive necessary conditions for dynamic optimization problems with random shocks, first in discrete time and then in continuous time.[3] We show that in the limiting process, the Lagrange multiplier approach is equivalent to the stochastic maximum principle, which has been around in various forms since Bismut (1975), see also Malliaris (1982). In the modern mathematical treatment of this theorem, see e.g. Pham (2009), the stochastic maximum principle is now presented in the context of backward stochastic differential equations, through which it has become much more accessible. The connection between the stochastic maximum principle and the Lagrange multiplier approach however does not seem to have been made, even in the mathematical literature. We acknowledge that our proof of the validity of the Lagrange method for continuous-time stochastic optimal control problem is tied to the validity of the stochastic maximum principle, we do not provide an independent proof. As such the stochastic maximum principle takes a central role in our investigation. However we find it more intuitive to start our investigation from the classical finite-dimensional Lagrange multiplier setup, and through appropriate modifications, generalizations, and specifically the continuous-time limit within a rigorous setup of functional and stochastic analysis, to eventually reach the stochastic maximum principle as an anchor point, rather than starting from the stochastic maximum principle and deriving the continuous time Lagrange multiplier equations from it. Nevertheless, both approaches are valid and any preference is a matter of taste.

In the economic literature, Chow (1992), Chow (1993), Kwan and Chow (1997), as well as Reiter (1996), have promoted the Lagrange multiplier approach to solve dynamic optimization problems in continuous time with random shocks generated through a Brownian motion, linking their approach to the classical Hamilton-Jacobi-Bellman approach to solve these problems. There are some gaps and limitations to these studies, which we aim to address and overcome in our current paper. Most importantly, none of these studies presents a formal Lagrange-type theorem, providing necessary conditions, derived in full mathematical rigor. Instead, for the discrete-time case, a Lagrange function is introduced and it is claimed that by differentiating the Lagrange function first order conditions are obtained that characterize the solution. It is ignored that their Lagrange function is actually defined on infinite dimensional spaces of stochastic processes and that taking derivatives

[3]For some background on standard terms of Functional Analysis, including a definition of Fréchet differentiability, we refer the reader to Treves (1967) and Ewald (2004).

on these spaces is a highly nontrivial matter. Furthermore, they rely on an argument, compare Chow (1993, pages 627-628), to separate individual terms within a sum which is part of an expected value, which from a mathematical point of view is not valid, even if it leads to the correct first order conditions. Additionally, their approach in discrete time relies on the fact that in their case the random shock is unaffected by either the control or state variable, a strong assumption that we overcome in our paper. For the continuous time case, neither Chow (1992), Chow (1993), Kwan and Chow (1997) or Reiter (1996) are connecting their approach to a formal Lagrange-type approach. Instead, their starting point is the Hamilton-Jacobi-Bellman equation, which after differentiating and defining the function $\lambda(x)$ as the first-order derivative of the value function of the underlying problem leads to an equation for $\lambda(x)$, to which they then refer as the Lagrange multiplier, see for example Chow (1993, page 624, equation (10)). But it remains unclear as to why their $\lambda(x)$ is in fact a Lagrange multiplier of the type considered in Theorem 1. The fact that $\lambda(x)$ denotes the marginal value of the constraint to the value function, i.e. $\lambda(x) := \frac{\partial V}{\partial x}(x)$ and that the same holds for the classical Lagrange multiplier does not provide sufficient argument. Indeed, in their approach, it is by definition only, while in the Lagrange multiplier approach, this is a non-trivial conclusion of the envelope theorem.

In this paper, we overcome many of the shortcomings and gaps of the Chow (1992) approach as indicated above. We show that a sufficiently general Lagrange multiplier theorem on Fréchet spaces can indeed be used to provide a necessary condition for a solution to a discrete-time optimization problem affected by random shocks. We allow random shocks to be affected by the state variable and control. Furthermore, by taking limits for the discretization step to approach zero, we obtain first-order conditions for the corresponding continuous time version of the problem. We then demonstrate that these conditions are in fact the very same conditions that are stated in the stochastic maximum principle, and as such have been around in various guises since Bismut (1975). The connections drawn between the stochastic maximum principle and the classical Lagrange approach are new and fundamental however and to the best of our knowledge have not been drawn within either the mathematical or economics literature.

One of the key contributions by Chow (1992), Chow (1993), Kwan and Chow (1997) as well as Reiter (1996) also lies in the potential for deriving numerical schemes that help solve dynamic optimization problems under uncertainty numerically and in approximation close to the corresponding steady state of the corresponding deterministic system. As Reiter (1996) however shows, there are nevertheless issues with the accuracy of the approximation, which is particularly prominent if a large shock shifts the state variables far away from the deterministic steady state. Our approach shares that strong potential for numerical applications. However, instead of expanding the Lagrange multiplier function λ up to a certain order in the state variable[4], we simulate the stochastic dynamics backward in time and regress the Lagrange multiplier, which by nature is a stochastic variable, backward onto powers of the state variable up to a certain degree. The methodology developed in our article naturally leads to this new method, which conceptually is related to the

[4]Chow (1993) and Reiter (1996) use a linear approximation

Longstaff and Schwartz least square Monte Carlo method as well as more modern methods for solving backward stochastic differential equations.[5] Instead of using simple powers in the regression step, we could also use more sophisticated polynomial functions or series of elementary functions, e.g. Hermite polynomials. However, in this paper, we simply want to present the potential of this approach, without aiming at optimizing it. The latter is left for future work. Nevertheless, our numerical examples show that even in a basic setup with linear regression on one factor only our method delivers excellent results for the benchmark optimal growth model presented in Reiter (1996). More generally, our computational results are important within the context where value function iteration methods are often presented as superior to the Lagrangian formulation on account of relative computational ease, we show that this is not necessarily correct.

While some parts of our paper point to gaps and limitations of Chow's Lagrangian approach, overall our results should be considered as an endorsement - even a celebration - of Chow's method, which as a whole is presented in Chow's excellent book Chow (1997), where it is amply demonstrated that it has many wide-ranging applications.

Beyond Chow, there are of course many other important contributions to the topic of stochastic optimal control in continuous time, many of these use duality, for example Ma et al (2019), and it would be very interesting to connect the Lagrange formalism explored in our paper to these. Further, Wörner et. al. (2015) propose an approximate relative value iteration algorithm based on piecewise-linear convex relative value function approximations, which would be interesting to investigate from the perspective of the Lagrange multiplier theorem and our proposed numerical scheme in section 4, which is partly Monte Carlo based. We leave both of these avenues for future research. In general terms, the importance of stochastic optimal control theory within the context of Operational Research has been emphasized in Neck (1984) and we see our contribution also in adding to this classical literature. Mathematical programming, Lagrange multipliers, and various adaptations to dynamic optimization have been discussed within the operations research literature in Hampshire and Williams (2014), but they do not cover continuous time and do not connect their approach to the stochastic maximum principle.

The remainder of the paper is organized as follows. In the following section 2, we present our main theoretical contributions, including the various Lagrange theorems as well as drawing the connection to the stochastic maximum principle. In section 3 we present two explicit examples that demonstrate the usefulness of our approach in theory. Further, in section 4 we illustrate how our approach can be used to derive new numerical schemes using backward Monte Carlo simulation and regression-based techniques in order to solve stochastic optimal control problems and benchmark its performance. Our main conclusions are summarized in section 5.

[5] Reiter (1999) has also explored linear regression-based methods for stochastic optimal control problems and Gobet et. al. (2005) present a regression-based Monte Carlo method to solve backward stochastic differential equations (without optimal control). Our approach is very different from Reiter but inspired by Gobet et al. (2005).

2 Lagrange Formalism for Stochastic Optimal Control

2.1 The Stochastic Maximum Principle

We consider a standard continuous time stochastic optimal control problem of the type

$$\max_{(\alpha_t)\in\mathcal{A}} \mathbb{E}\left(\int_0^T f(t,X_t,\alpha_t)dt + g(X_T)\right) \tag{4}$$

subject to the dynamic constraint

$$dX_t = b(t,X_t,\alpha_t)dt + \sigma(t,X_t,\alpha_t)dW_t. \tag{5}$$

For simplicity, we assume that the state variable (X_t) and Brownian motion (W_t) are one-dimensional, but all that follows can be adapted to the multi-dimensional case using vector-matrix notation. Measurability is defined with respect to the Brownian filtration $(\mathcal{F}_t)$ and the set of admissible controls $\mathcal{A}$ is the set of progressively measurable stochastic processes taking values in the set A. We assume that the functions $f, b, \sigma : [0,T]\times\mathbb{R}\times A \to \mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ are continuously differentiable with bounded derivatives.[6]

Perhaps the most sophisticated method developed to solve problems of this type is the stochastic maximum principle. We follow Pham (2009) in his exposition. For this we define the Hamiltonian[7]

$$\mathcal{H}(t,x,a,y,z) = b(t,x,a)y + \sigma(t,x,a)z + f(t,x,a). \tag{6}$$

We denote derivatives with lower indices, i.e. $\mathcal{H}_x, f_x, f_a, g_x, b_x, b_a, \sigma_x, \sigma_a$, and higher derivatives with multiple lower indices. The following Theorem characterizes the solution of problem (4)-(5) as maximizing the Hamiltonian along the solution of a backward stochastic differential equation (BSDE).[8] The form in which the Theorem is presented is due to Pham (2009) (Theorem 6.4.6), but (at least for special cases) the Theorem can be traced back to Bismut (1975).

Theorem 2. *Assume that $(\alpha_t^*)\in\mathcal{A}$ and the pair $((Y_t^*),(Z_t^*))$ is a solution to the BSDE*

$$\begin{aligned} -dY_t &= \mathcal{H}_x(t,X_t^*,\alpha_t^*,Y_t,Z_t)dt - Z_t dW_t, && (7)\\ Y_T &= g_x(X_T^*), && (8) \end{aligned}$$

such that

$$\mathcal{H}(t,X_t^*,\alpha_t^*,Y_t^*,Z_t^*) = \max_{a\in A}\mathcal{H}(t,X_t^*,a,Y_t^*,Z_t^*) \tag{9}$$

[6] These restrictions can be relaxed but this requires more sophisticated functional analysis and would require case-by-case consideration.

[7] This is equivalent to Pham (2009) equation (6.24) noticing that Pham uses a multi-dimensional setup, where the first component of the multi-dimensional process (X_t) can be chosen as time t. We use a one-dimensional notation for (X_t) but include the variable t explicitly.

[8] Including the dynamics (5) in the formal setup, it becomes a forward-backward stochastic differential equation (FBSDE).

for $0 \le t \le T$ almost surely, where X_t^* is the solution of (5) under the control (α_t^*). If the function

$$(x, a) \mapsto \mathcal{H}(t, x, a, Y_t^*, Z_t^*) \tag{10}$$

is concave for all $t \in [0, T]$ a.s., then (α_t^*) is the solution of the stochastic optimal control problem (4)-(5).

Under the assumption of an interior maximum in (9), we have that

$$b_a(t, X_t^*, \alpha_t^*)Y_t^* + \sigma_a(t, X_t^*, \alpha_t^*)Z_t^* + f_a(t, X_t^*, \alpha_t^*) = 0. \tag{11}$$

Further we have that

$$-dY_t^* = [b_x(t, X_t^*, \alpha_t^*)Y_t^* + \sigma_x(t, X_t^*, \alpha_t^*)Z_t^* + f_x(t, X_t^*, \alpha_t^*)]\, dt - Z_t^* dW_t, \tag{12}$$

with $Y_T^* = g_x(X_T^*)$. Equations (11) and (12) will be important to link the process (Y_t^*) to the continuous time Lagrangian. Before that, further identification of (Y_t^*) and (Z_t^*) is useful. In fact, introducing the value function of the problem (4)-(5) as

$$v(t, x) = \mathbb{E}\left(\int_t^T f(s, X_s^*, \alpha_s^*)ds + g(X_T^*)\middle| X_t^* = x\right), \tag{13}$$

it can be shown that if v is sufficiently smooth, i.e. $v \in C^{1,3}$, then

$$\begin{aligned} Y_t^* &= v_x(t, X_t^*) && (14)\\ Z_t^* &= v_{xx}(t, X_t^*)\sigma(t, X_t^*, \alpha_t^*), && (15) \end{aligned}$$

see Pham (2009) equation (6.35).

2.2 Infinite Dimensional Lagrange Theorem and Discretization

Let us now present the Lagrangian approach to solving problem (4)-(5). To do this, we first discretize problem (4)-(5). We will later consider the continuous time limit. We assume that $0 = t_0 < t_1 < ... < t_n = T$ is an equidistant partition and define $\Delta = t_{i+1} - t_i$. We denote with

$$\Delta W_i = W_{t_{i+1}} - W_{t_i} \tag{16}$$

the increment of the Brownian motion (W_t) over the corresponding interval. We further let $\mathcal{F}_i = \mathcal{F}_{t_i}$ and $M^i = L^2(\Omega, \mathcal{F}_i)$ be the space of square integrable $\mathcal{F}_i$ measurable random variables on Ω. Further we define

$$\begin{aligned} \mathbb{M}_n &= M^0 \times M^1 \times ... \times M^n && (17)\\ \mathbb{M}_n^- &= M^0 \times M^1 \times ... \times M^{n-1}. && (18) \end{aligned}$$

We then consider the following constrained optimization problem on $\mathbb{M}_n \times \mathbb{M}_n^-$:

$$
\begin{aligned}
\max_{((X_i),(\alpha_i))} \mathbb{E}\left(\sum_{i=0}^{n-1} f(i\Delta, X_i, \alpha_i)\Delta + g(X_n)\right) && (19)\\
s.t.\ X_0 &= x \text{ and for } i = 0, ..., n-1 && (20)\\
X_{i+1} - X_i &= b(i\Delta, X_i, \alpha_i)\Delta + \sigma(i\Delta, X_i, \alpha_i)\Delta W_i. && (21)
\end{aligned}
$$

We will apply the Lagrange theorem for real Banach spaces, Proposition 1 in Zeidler (1995), page 270. Using the same notation, we define the two functions F and G as follows:

$$
\begin{aligned}
F : \mathbb{M}_n \times \mathbb{M}_n^- &\to \mathbb{R} && (22)\\
((X_i), (\alpha_i)) &\mapsto \mathbb{E}\left(\sum_{i=0}^{n-1} f(i\Delta, X_i, \alpha_i)\Delta + g(X_n)\right) && (23)
\end{aligned}
$$

and

$$
\begin{aligned}
& G : \mathbb{M}_n \times \mathbb{M}_n^- \to \mathbb{M}_n && (24)\\
& ((X_i), (\alpha_i)) \mapsto \begin{pmatrix} x - X_0 \\ X_0 + b(0\Delta, X_0, \alpha_0)\Delta + \sigma(0\Delta, X_0, \alpha_0)\Delta W_0 - X_1 \\ \vdots \\ X_{n-1} + b((n-1)\Delta, X_{n-1}, \alpha_{n-1})\Delta + \sigma((n-1)\Delta, X_{n-1}, \alpha_{n-1})\Delta W_{n-1} - X_n \end{pmatrix}.
\end{aligned}
$$

The constrained optimization problem (19)-(21) can then be expressed as

$$
\begin{aligned}
& \max F((X_i), (\alpha_i))) && (25)\\
s.t.\quad & G((X_i), (\alpha_i)) = 0. && (26)
\end{aligned}
$$

Both F and G are Fréchet differentiable and their respective derivatives at a chosen point $((X_i^*), (\alpha_i^*)) \in \mathbb{M}_n \times \mathbb{M}_n^-$ are given as follows:[9]

$$
\begin{aligned}
DF((X_i^*), (\alpha_i^*)) : \mathbb{M}_n \times \mathbb{M}_n^- &\to \mathbb{R} && (27)\\
((\xi_i), (\beta_i)) &\mapsto \mathbb{E}\left(\sum_{i=0}^{n-1} f_x(i\Delta, X_i^*, \alpha_i^*)\Delta \cdot \xi_i + g_x(X_n^*) \cdot \xi_n \right.\\
&\qquad \left. + \sum_{i=0}^{n-1} f_a(i\Delta, X_i^*, \alpha_i^*)\Delta \cdot \beta_i\right)
\end{aligned}
$$

[9] For this conclusion, note that for any continuously differentiable function $\epsilon : \mathbb{R} \to \mathbb{R}$ with bounded derivatives the induced function $X \mapsto \epsilon \circ X$ which maps M^i onto itself by composition has Fréchet differential $D\epsilon(X)Y = \epsilon'(X) \cdot Y$, where the latter product is the point-wise product of a bounded function and a function in M^i and $\epsilon'(x)$ is the one-dimensional derivative of $\epsilon(x)$ in the standard calculus way. This can be verified by applying the definition of the Fréchet derivative and the derivative from standard calculus.

and

$$DG((X_i^*), (\alpha_i^*)) : \mathbb{M}_n \times \mathbb{M}_n^- \to \mathbb{M}_n \tag{28}$$

$$((\xi_i), (\beta_i)) \mapsto$$

$$\begin{pmatrix} -1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 1 + b_x\Delta + \sigma_x\Delta W & -1 & 0 & \vdots & b_u\Delta + \sigma_u\Delta W & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & 0 & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots \quad 0 & \vdots & & \ddots & 0 \\ 0 & \cdots & 1 + b_x\Delta + \sigma_x\Delta W & -1 & 0 & \cdots & 0 & b_u\Delta + \sigma_u\Delta W \end{pmatrix} \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_n \\ \beta_0 \\ \vdots \\ \beta_{n-1} \end{pmatrix}$$

In the matrix above the expressions $1+b_x\Delta+\sigma_x\Delta W$ and $b_u\Delta+\sigma_u\Delta W$ are evaluated at $(i\Delta, X_i^*, \alpha_i^*)$ according to their position in the matrix. Matrices and vectors are multiplied in linear algebra fashion and products within L^2 spaces are point-wise. As the matrix representing DG has full rank, Proposition 1 on page 270 in Zeidler (1995) implies that if $((X_i^*), (\alpha_i^*))$ is a solution to (25)-(26), then there exists $\lambda \in \mathbb{M}_n^*$ such that

$$DF + \lambda \circ DG = 0. \tag{29}$$

Here $\mathbb{M}_n^*$ denotes the dual space of $\mathbb{M}_n$, however since only L^2 spaces are involved, we have $\mathbb{M}_n^* \cong \mathbb{M}_n$, via the natural isomorphism which identifies $(\lambda_i) \in \mathbb{M}_n$ with the functional

$$(\zeta_i) \mapsto \mathbb{E}\left(\sum_{i=0}^{n} \lambda_i \cdot \zeta_i\right) \tag{30}$$

for all $(\zeta_i) \in \mathbb{M}_n$. Equation (29) is to be understood as the functional on the left hand side being identical zero on $\mathbb{M}_n \times \mathbb{M}_n^-$, where $\lambda \circ DG$ indicates the composition of DG and $\lambda = (\lambda_i)$. Using this identification, (29) is equivalent to

$$\begin{aligned} &\mathbb{E}\left(\sum_{i=0}^{n-1} f_x(i\Delta, X_i^*, \alpha_i^*)\Delta \cdot \xi_i + g_x(X_n^*) \cdot \xi_n + \sum_{i=0}^{n-1} f_a(i\Delta, X_i^*, \alpha_i^*)\Delta \cdot \beta_i\right) \\ &\quad + \mathbb{E}\left(\lambda_0(-\xi_0) + \lambda_1\left[\xi_0(1 + b_x\Delta + \sigma_x\Delta W_1) - \xi_1 + \beta_0(b_a\Delta + \sigma_a\Delta W_1)\right]\right. \\ &\qquad\qquad \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ &\quad \left.\lambda_n\left[\xi_{n-1}(1 + b_x\Delta + \sigma_x\Delta W_{n-1}) - \xi_n + \beta_{n-1}(b_a\Delta + \sigma_a\Delta W_{n-1})\right]\right) = 0 \end{aligned} \tag{31}$$

for all $((\xi_i),(\beta_i)) \in \mathbb{M}_n \times \mathbb{M}_n^-$. Collecting terms shows that (31) is equivalent to

$$\begin{aligned} &\mathbb{E}\left(\sum_{i=0}^{n-1} f_x(i\Delta, X_i^*, \alpha_i^*)\Delta \cdot \xi_i + g_x(X_n^*)\cdot \xi_n + \sum_{i=0}^{n-1} f_a(i\Delta, X_i^*, \alpha_i^*)\Delta \cdot \beta_i\right) \\ &\quad + \mathbb{E}\left(\sum_{i=0}^{n-1}(\lambda_{i+1}-\lambda_i)\xi_i + \lambda_{i+1}(b_x\Delta + \sigma_x \Delta W_i)\xi_i - \lambda_n \xi_n \right. \\ &\quad\quad \left. \sum_{i=0}^{n-1} \lambda_{i+1}\beta_i(b_a\Delta + \sigma_a \Delta W_i)\right) = 0. \end{aligned} \tag{32}$$

The terms $\lambda_{i+1}(b_x\Delta + \sigma_x\Delta W_i)$ in the sum above can be replaced by $(\lambda_{i+1}-\lambda_i)(b_x\Delta+\sigma_x\Delta W_i) + \lambda_i(b_x\Delta + \sigma_x\Delta W_i)$ and similar for $\lambda_{i+1}(b_a\Delta + \sigma_a\Delta W_i)$. This is necessary to identify the continuous time limits in the context of stochastic calculus.[10] Using this trick, we obtain

$$\begin{aligned} &\mathbb{E}\left(\sum_{i=0}^{n-1} f_x(i\Delta, X_i^*, \alpha_i^*)\Delta \cdot \xi_i + g_x(X_n^*)\cdot \xi_n + \sum_{i=0}^{n-1} f_a(i\Delta, X_i^*, \alpha_i^*)\Delta \cdot \beta_i\right) \\ &+\mathbb{E}\left(\sum_{i=0}^{n-1} \xi_i\left[(\lambda_{i+1}-\lambda_i) + (\lambda_{i+1}-\lambda_i)(b_x\Delta + \sigma_x\Delta W_i) + \lambda_i(b_x\Delta + \sigma_x\Delta W_i)\right]\right. \\ &\quad \left. -\lambda_n\xi_n + \sum_{i=0}^{n-1}\beta_i\left[(\lambda_{i+1}-\lambda_i)(b_a\Delta + \sigma_a\Delta W_i) + \lambda_i(b_a\Delta + \sigma_u\Delta W_i)\right]\right) = 0 \end{aligned} \tag{33}$$

for all $((\xi_i),(\beta_i)) \in \mathbb{M}_n \times \mathbb{M}_n^-$. As $\xi_n \in M^n$ can be chosen independently, we immediately obtain that

$$\lambda_n = g_x(X_n^*). \tag{34}$$

[10] The reason for this is that the stochastic Itô integral (in the limit) is defined through evaluating the integrand on the left-hand side, even though other types of stochastic integrals exist, but are not common in economic modeling.

2.3 From Discrete Time to Continuous Time

We now let Δ go to zero, and assume that the corresponding sequence of (λ_i)'s converges to a progressively measurable Itô process[11] in the continuous time limit.[12] Then we obtain that

$$\begin{aligned} &\mathbb{E}\left(\int_0^T \xi_t f_x(t, X_t^*, \alpha_t^*)dt + \int_0^T \beta_t f_a(t, X_t^*, \alpha_t^*)dt\right) \\ &+\mathbb{E}\left(\int_0^T \xi_t \left[d\lambda_t + d\lambda_t(b_x dt + \sigma_x dW_t) + \lambda_t(b_x dt + \sigma_x dW_t)\right]\right. \\ &\left. + \int_0^T \beta_t \left[d\lambda_t(b_a dt + \sigma_a dW_t) + \lambda_t(b_a dt + \sigma_a dW_t)\right]\right) = 0, \end{aligned} \tag{35}$$

as well as

$$\lambda_T = g_x(X_T^*), \tag{36}$$

with $b_x, b_a, \sigma_x, \sigma_a$ evaluated at (t, X_t^*, α_t^*). Itô calculus implies that differential products of the type $d\lambda_t dt$ are all zero[13] and integrals with regards to dW have zero expectation. Hence (35) reduces to

$$\begin{aligned} &\mathbb{E}\left(\int_0^T \xi_t f_x(t, X_t^*, \alpha_t^*)dt + \int_0^T \beta_t f_a(t, X_t^*, \alpha_t^*)dt\right) + \\ &\mathbb{E}\left(\int_0^T \xi_t \left[d\lambda_t + \sigma_x d\lambda_t dW_t) + \lambda_t b_x dt\right] + \int_0^T \beta_t \left[\sigma_a d\lambda_t dW_t + \lambda_t b_a dt\right]\right) \quad = \quad 0, \end{aligned} \tag{37}$$

for all $(\xi_t), (\beta_t) \in L^2(\Omega \times [0, T])$. As (ξ_t) and (β_t) can be chosen independently from each other, we have that (37) is equivalent to the following two equalities:

$$\mathbb{E}\left(\int_0^T \xi_t \left[f_x(t, X_t^*, \alpha_t^*)dt + d\lambda_t + \sigma_x(t, X_t^*, \alpha_t^*)d\lambda_t dW_t + \lambda_t b_x(t, X_t^*, \alpha_t^*)dt\right]\right) = 0 \tag{38}$$

$$\mathbb{E}\left(\int_0^T \beta_t \left[f_a(t, X_t^*, \alpha_t^*)dt + \sigma_a(t, X_t^*, \alpha_t^*)d\lambda_t dW + \lambda_t b_a(t, X_t^*, \alpha_t^*)dt\right]\right) = 0 \tag{39}$$

We refer to the solution (λ_t) of (38) and (39) as the continuous time Lagrange multiplier. As (ξ_t) and (β_t) can be chosen as arbitrary $(\mathcal{F}_t)$ adapted L^2-processes, the latter two equations can be formally written as[14]

---

[11] We refer to Pham (2009) Definition 1.2.11 for the definition of an Itô process and note that the class of Itô processes is contained in the class of semi-martingales and that therefore the relevant stochastic integrals in (35) exist and in fact correspond to the continuous time limit of (33). Quadratic co-variations such as $d\lambda_t dW_t$ are well defined under this assumption too and represent suitable integrators.

[12] Some literature exists about approximation of continuous time stochastic control problems by discretization and convergence of the discretized solution. We do not delve into this literature. We will later show that the continuous time characterization of the Lagrange problem has a solution in form of the solution to the BSDE that is attached to the stochastic maximum principle. Hence this assumption is more for motivation than it is necessary to establish the general principle.

[13] Intuitively, within the Itô calculus and in approximation $dW \simeq \sqrt{dt} \simeq \sqrt{\Delta}$, this result follows from $\lim_{\Delta\to 0} \sum_{i=1}^{T/\Delta} \Delta^2 = \lim_{\Delta\to 0} \sum_{i=1}^{T/\Delta} \Delta^{3/2} = 0$.

[14] Note that possible choices for the processes $(\xi_s), (\beta_s)$ include for example $\xi_s := \mathbb{1}_{s\geq t} X^t$ and $\beta_s := \mathbb{1}_{s\geq t} Y^t$ for arbitrary $\mathcal{F}_t$ measurable random variables X^t and Y^t.

$$-\frac{\mathbb{E}_t(d\lambda_t)}{dt} = f_x(t, X_t^*, \alpha_t^*) + \sigma_x(t, X_t^*, \alpha_t^*)\frac{d\lambda_t dW_t}{dt} + \lambda_t b_x(t, X_t^*, \alpha_t^*) \tag{40}$$

$$0 = f_a(t, X_t^*, \alpha_t^*) + \sigma_a(t, X_t^*, \alpha_t^*)\frac{d\lambda_t dW}{dt} + \lambda_t b_a(t, X_t^*, \alpha_t^*), \tag{41}$$

where $\mathbb{E}_t(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_t)$. Equations (40) and (41) are the key equations of our Lagrangian approach in continuous time. They can be used to identify the optimal controls and the Lagrangian λ as we show later in our examples. In addition, equations (40) and (41) open a gateway for deriving numerical schemes as we will demonstrate in section 4. To formally link these equations to a Lagrangian type theorem we revert back to the discretized version of the problem and formally introduce the Lagrange function

$$\mathcal{L} = F + \lambda \circ G \tag{42}$$

for the discretized problem. This translates into

$$\mathcal{L}((X_i), (\alpha_i), (\lambda_i)) = \mathbb{E}\Bigg(\sum_{i=0}^{n-1} f(i\Delta, X_i, \alpha_i)\Delta + g(X_n) - \lambda_0(X_0 - x)$$
$$- \sum_{i=1}^{n-1} \lambda_i \left[(X_i - X_{i-1}) - b((i-1), X_{i-1}, \alpha_{i-1})\Delta - \sigma((i-1), X_{i-1}, \alpha_{i-1})\Delta W_i\right]\Bigg).$$

Again, the products of λ_i with $(X_i - X_{i-1})$ and ΔW_i are problematic from a stochastic calculus point of view, which is why we rewrite $\lambda_i = \lambda_{i-1} + (\lambda_i - \lambda_{i-1})$ as before. Then moving to the continuous time limit for $\Delta \to 0$ we obtain for the continuous time Lagrange function[15]

$$\begin{aligned}\mathcal{L}((X_t), (\alpha_t), (\lambda_t)) &= \mathbb{E}\Bigg(\int_0^T f(t, X_t, \alpha_t)dt + g(X_T) \\ &\quad - \int_0^T \lambda_t(dX_t - b(t, X_t, \alpha_t)dt - \sigma(t, X_t, \alpha_t)dW_t) \\ &\quad - \int_0^T d\lambda_t dX_t + \int_0^T \sigma(t, X_t, \alpha_t)d\lambda_t dW_t\Bigg).\end{aligned} \tag{44}$$

Finally, applying the Itô partial integration rule, the latter is equivalent to

[15] We may alternatively define the Lagrange function $\tilde{\mathcal{L}}$ via

$$\begin{aligned}\tilde{\mathcal{L}} &= \mathbb{E}\Bigg(\int_0^T f(t, X_t, \alpha_t)dt + g(X_T) \\ &\quad - \int_0^T \lambda_t(dX_t - b(t, X_t, \alpha_t)dt - \sigma(t, X_t, \alpha_t)dW_t),\end{aligned} \tag{43}$$

which at first glance seems more natural and more closely tied to the standard Lagrange formalism. The previous analysis shows however, that by strictly following the standard Lagrange formalism and making the transition to continuous time stochastic dynamics, in the limit we obtain $\mathcal{L}$ and not $\tilde{\mathcal{L}}$.

$$\begin{aligned}\mathcal{L} &= \mathbb{E}\Big(\int_0^T f(t,X_t,\alpha_t)dt + g(X_T) \qquad (45)\\ &\quad + \int_0^T [X_t - x - \int_0^t b(s,X_s,\alpha_s)ds - \sigma(s,X_s,\alpha_s)dW_s]d\lambda_t,\end{aligned}$$

which resembles a more classical textbook Lagrangian function, where the Lagrangian is now represented by the differential $d\lambda_t$ and the constraint is equation (5) with initial condition $X_0 = x$. We now come to our main theorem:

Theorem 3. *1. Let (α_t^*) be an admissible control and (X_t^*) the corresponding state process. Assume there exists a progressively measurable process (λ_t^*) such that*

$$\frac{\partial \mathcal{L}}{\partial X}((X_t^*),(\alpha_t^*),(\lambda_t^*)) = 0 \qquad (46)$$

$$\frac{\partial \mathcal{L}}{\partial \alpha}((X_t^*),(\alpha_t^*),(\lambda_t^*)) = 0. \qquad (47)$$

Further assume that the function

$$(x,a) \mapsto b(t,x,a)\cdot\lambda_t^* + \sigma(t,x,a)\cdot\frac{d\lambda_t^* dW_t}{dt} + f(t,x,a) \qquad (48)$$

is concave for all $t \in [0,T]$ a.s.. Then (α_t^*) is the optimal control.[16]

2. On the other hand, if (α_t^*) is a solution to the problem (4)-(5) and (48) holds, then there exists a progressively measurable Itô process (λ_t^*) such that (46)-(47) holds.

Proof. We have established above, that (46) and (47) are equivalent to (36), (38) and (39). Choosing processes (ξ_t^u) in (38) where $\xi_s^u = 1$ for all $s > u$ and otherwise unrestricted (but progressively measurable) shows that the term in the square brackets in (38) defines a martingale. The martingale representation theorem then establishes the existence of a process (Z_t^*), such that

$$d\lambda_t^* + \left(b_x(t,X_t^*,\alpha_t^*) + \sigma_x(t,X_t^*,\alpha_t^*)d\lambda_t^* dW_t + f_x(t,X_t^*,\alpha_t^*)\right)dt = Z_t^* dW_t. \qquad (49)$$

Using that the term in brackets on the left hand side is of bounded variation, we then obtain

$$d\lambda_t^* dW_t = Z_t^* dt. \qquad (50)$$

Substituting the latter in (49) and reordering shows that $((\lambda_t^*),(Z_t^*))$ is a solution of the BSDE (7)-(8). Then (48) implies (10) and (39) implies (9). The stochastic maximum principle then establishes that (α_t^*) is the optimal control.

[16] Carrying out the differentiation one needs to use that $\frac{\partial dX}{\partial X}$ is the functional which gives $\left(\int_0^T \lambda_t \left(\frac{\partial dX}{\partial X}\right)_t\right)(\xi_t) = \int_0^T \xi_t d\lambda_t$. Equations (46)-(47) have to be considered in the functional sense. However, equation (48) refers to concavity of deterministic functions depending on $(x,a) \in \mathbb{R}^2$ for fixed states $\omega \in \Omega$ and fixed time t, analog to (10).

On the other hand, let (α_t^*) be a solution to the problem (4)-(5). We then define $\lambda_t^* = v_x(t, X_t^*)$ and $Z_t^* = v_{xx}(t, X_t^*)\sigma(t, X_t^*, \alpha_t^*)$. Pham (2009) (Theorem 6.4.7, page 151) shows that the pair $((\lambda_t^*), (Z_t^*))$ is a solution of the BSDE (7) – (8). Then using equation (12) (which is equivalent to (7)), we find that

$$d\lambda_t^* dW_t = v_{xx}(t, X_t^*)\sigma(t, X_t^*, \alpha_t^*)dt. \tag{51}$$

We substitute this for $d\lambda_t dW$ and also substitute (12) for $d\lambda_t$ on the left hand side of (38). Then, noticing that the expectation of integrals with respect to dW are zero shows that (38) indeed holds. Similarly, substitution of (51) into (39) and using (48) shows that (39) holds, note equation (6.34) in Pham (2009). □

We conclude this section with a short note on duality. The problem presented in Theorem 1 has a dual problem and a link between the two solutions of the primal and dual problem is presented through an appropriate duality theorem. Such a duality theorem for stochastic optimal control problems can be found in Chapter 6 of Karatzas and Shreve (1998) Proposition 5.1 and Theorem 5.3. We expect that our stochastic Lagrange formalism connects duality in the finite-dimensional context of the introduction with the framework presented in Karatzas and Shreve (1998). However to formally carry out this work would be material for an independent research artcile and we postpone this to future work.

3 Examples

3.1 Stochastic Linear Regulator

We start this section by considering the stochastic linear regulator. This example is relevant to a number of applications in economics and finance and takes center stage in many textbooks on dynamic optimization in economics including the classic textbook Sargent (1987). To contrast the Lagrangian approach with the mainstream Hamilton-Jacobi-Bellman (HJB) approach, we start with the latter. The HJB approach takes its starting point from the value function $v(t, x)$ defined in (13) and the corresponding HJB equation

$$v_t(t, x) + \max_{(\alpha_t)\in\mathcal{A}} \left\{ f(t, x, \alpha) + v_x(t, x)b(t, x, \alpha) + \frac{1}{2}v_{xx}(t, x)\sigma(t, x, \alpha)^2 \right\} = 0, \tag{52}$$

with terminal condition $v(T, x) = g(x)$. For the particular problem of the stochastic linear regulator we have

$$\begin{aligned} v(t, x) &= \min_{(\alpha_t)\in\mathcal{A}} \mathbb{E}^x \left(\int_0^T (X_t^2 + \theta\alpha_t^2)dt + \Gamma X_T^2 \right), && (53) \\ dX_t &= \alpha_t dt + \sigma dW_t. && (54) \end{aligned}$$

The notation introduced in section 2 previously considers the maximum, but one can easily convert the two via the relationship $\min(obj) = \max(-obj)$ and hence with the same notation as in section 2 we have

$$
\begin{aligned}
f(t,x,\alpha) &= -(x^2+\theta\alpha^2) \text{ with } f_x = -2x \text{ and } f_\alpha = -2\theta\alpha, && (55)\\
g(x) &= -\Gamma x^2, \text{ with } g_x = -2\Gamma x, && (56)\\
b(t,x,\alpha) &= \alpha, \text{ with } b_x = 0 \text{ and } b_\alpha = 1, && (57)\\
\sigma(t,x,\alpha) &= \sigma, \text{ with } \sigma_x = 0 \text{ and } \sigma_\alpha = 0. && (58)
\end{aligned}
$$

The maximization problem in the HJB equation (52) then becomes

$$\max_\alpha \left\{ -(x^2+\theta\alpha^2) + v_x \cdot a + \frac{1}{2} v_{xx}(t,x)\sigma^2 \right\}, \tag{59}$$

with first order condition

$$-2\theta\alpha + V_x = 0 \Rightarrow \alpha = \frac{1}{2\theta} v_x. \tag{60}$$

Substitution into the HJB equation and collecting terms gives

$$v_t - x^2 + \frac{1}{4\theta} v_x^2 + \frac{1}{2} v_{xx}\sigma^2 = 0, \tag{61}$$

with boundary condition $v(T,x) = -\Gamma x^2$. This is a non-linear partial differential equation and not straightforward to solve. However, experience guides and a sophisticated guess of the functional form

$$v(t,x) = h(t)x^2 + g(t), \tag{62}$$

with suitable differentiable function $h(t)$ and $g(t)$ will lead to its solution. In fact, substituting (62) in (82) and collecting terms in powers of x gives

$$\left\{ h'(t) - 1 + \frac{1}{\theta} h(t)^2 \right\} x^2 + \left\{ g'(t) + h(t)\sigma^2 \right\} = 0, \tag{63}$$

with boundary conditions $h(T) = -\Gamma$ and $g(T) = 0$ derived from the corresponding boundary condition of $v(t,x)$. As the variable x can be varied freely, this yields a system of two ordinary differential equations

$$
\begin{aligned}
h'(t) + \frac{1}{\theta} h(t)^2 &= 1 && (64)\\
g'(t) + h(t)\sigma^2 &= 0. && (65)
\end{aligned}
$$

This system presents a form of a Riccati differential equation and can fortunately be solved. The solution for $h(t)$ is

$$h(t) = -\sqrt{\Gamma} \left(\frac{1 + \beta e^{\frac{2t}{\sqrt{\theta}}}}{1 - \beta e^{\frac{2T}{\sqrt{\theta}}}} \right), \text{ with } \beta = \frac{\Gamma - \sqrt{\theta}}{\Gamma + \sqrt{\theta}} \cdot e^{\frac{-2T}{\sqrt{\theta}}}. \tag{66}$$

The solution for $g(t)$ can be obtained by integration

$$g(t) = \sigma^2 \int_t^T h(s)ds = -\sqrt{\Gamma}\left\{(T-t) + \frac{1}{2}\frac{\beta\cdot\left(e^{\frac{2t}{\sqrt{\theta}}} - e^{\frac{2T}{\sqrt{\theta}}}\right)\sqrt{\theta}}{\beta e^{\frac{2T}{\sqrt{\theta}}} - 1}\right\}. \tag{67}$$

With this and (62) we have obtained the value function of the stochastic linear regulator problem. The optimal policy is derived from (60) and (62) as

$$\alpha_t = \frac{1}{\theta}h(t)X_t. \tag{68}$$

Let us now solve the same problem with the Lagrangian method. The starting point for the Lagrange method are equations (40) and (41) which involve the Lagrange multiplier λ_t. Using (55)-(58) we obtain

$$\frac{E_t(d\lambda_t)}{dt} = 2X_t \text{ and } \alpha_t = \frac{1}{2\theta}\lambda_t. \tag{69}$$

The optimal policy α_t is consistent with those derived in (60), but note that this is obtained in the initial step of the Lagrangian approach already. In order to identify the Lagrange multiplier λ_t we make a sophisticated guess $\lambda_t = \epsilon(t)X_t$.[17] Using the Itô product rule we obtain

$$\begin{aligned} d\lambda_t &= \epsilon'(t)X_t dt + \epsilon(t)dX_t \\ &= \left\{\epsilon'(t) + \frac{1}{2\theta}\epsilon(t)^2\right\}X_t dt + \epsilon(t)\sigma dW_t. \end{aligned} \tag{70}$$

Comparing (70) with the first part of (69) we obtain

$$\epsilon'(t) + \frac{1}{2\theta}\epsilon(t)^2 = 2 \text{ with } \epsilon(t) = -2\Gamma. \tag{71}$$

Comparing this with (64) one obtains immediately that $\epsilon(t) = 2\cdot h(t)$ and one obtains a closed form for $\epsilon(t)$ from (66). Without our previous work relating to the HJB approach, this work would of course also need to be carried out in the Lagrangian approach. However, note how quickly we arrived at the point where we have both the Lagrangian and the optimal policy

$$\alpha_t = \frac{\epsilon(t)}{2\theta}X_t \tag{72}$$

fully determined. There is only one ODE (71) in the Lagrange approach as opposed to the two (coupled) ODE's (64) and (65) in the HJB approach and we never have to make use of second-order derivatives. On the basis of the above, at least for the stochastic linear regulator, the Lagrange approach appears as much more straightforward.

[17] This sophisticated guess is at the same level of abstraction as the one that led to (62).

3.2 Neoclassical Growth

Let us now consider another example, the neoclassical growth model, which has also been discussed in Kwan and Chow (1997). Here

$$\max \mathbb{E}\left(\int_0^\infty e^{-\rho t} u(c_t) dt\right) \tag{73}$$

$$dk_t = (f(k_t) - c_t)dt + \sigma k_t dW_t. \tag{74}$$

Equations (40) and (41) imply

$$-\frac{\mathbb{E}_t d\lambda_t}{dt} = \lambda_t f'(k_t) + \sigma \frac{d\lambda_t dW_t}{dt} \tag{75}$$

$$\lambda_t = e^{-\rho t} u'(c_t). \tag{76}$$

For convenience we define $\tilde{\lambda}_t = e^{\rho t}\lambda_t$ and hence $\tilde{\lambda}_t = u'(c_t)$. Because the problem (73) and (74) is time homogenous and Markovian we can then conclude that $\tilde{\lambda}_t = \tilde{\lambda}(k_t)$ with $\tilde{\lambda}(k)$ a sufficiently smooth function of k. We then obtain from the Itô formula that

$$d\lambda_t = \rho e^{\rho t}\tilde{\lambda}(k_t)dt + e^{\rho t}\tilde{\lambda}'(k_t)dk_t + \frac{1}{2}e^{\rho t}\tilde{\lambda}''(k_t)(dk_t)^2, \tag{77}$$

and hence

$$d\lambda_t dW_t = e^{\rho t}\tilde{\lambda}'(k_t)\sigma k_t dt. \tag{78}$$

Therefore equation (75) becomes

$$\rho\tilde{\lambda}(k_t) + \tilde{\lambda}'(k_t)(f(k_t) - c(k_t)) + \frac{1}{2}\tilde{\lambda}''(k_t)\sigma^2 k_t^2 + \tilde{\lambda}(k_t)f'(k_t) + \tilde{\lambda}'(k_t)\sigma k_t = 0. \tag{79}$$

We denote with $I(x) = (u')^{-1}(x)$ the inverse function of $u'(x)$. Then $c(k) = I(\tilde{\lambda}(k))$ and we obtain

$$\tilde{\lambda}(k)(\rho + f'(k)) + \tilde{\lambda}'(k)(f(k) - I(\tilde{\lambda}(k)) + \sigma k) + \frac{1}{2}\tilde{\lambda}''(k)\sigma^2 k^2 = 0. \tag{80}$$

In general this is a second order non-linear ordinary differential equation. Under some specific assumptions about the utility function $u(c)$ and the production function $f(k)$ analytic solutions can be obtained. Let us therefore assume that

$$u(c) = \frac{1}{\gamma}c^\gamma \text{ and } f(k) = k^\alpha. \tag{81}$$

Then obviously $I(x) = x^{\frac{1}{\gamma-1}}$ and $f'(k) = \alpha k^{\alpha-1}$ and equation (59) becomes

$$\tilde{\lambda}(k)(\rho+\alpha k^{\alpha-1})+\tilde{\lambda}'(k)\left(k^{\alpha}-\tilde{\lambda}(k)^{\frac{1}{\gamma-1}}+\sigma k\right)+\frac{1}{2}\tilde{\lambda}''(k)\sigma^2 k^2=0. \tag{82}$$

This ordinary differential equation still does not permit an explicit solution unless $\gamma = 1-\alpha$. In the latter case we try the following functional form for $\tilde{\lambda}(k)$:

$$\tilde{\lambda}(k)=mk^{-\alpha}. \tag{83}$$

Substitution into (82) gives

$$mk^{-\alpha}(\rho+\alpha k^{\alpha-1})-m\alpha k^{-(\alpha+1)}\left(k^{\alpha}-m^{-\frac{1}{\alpha}}k+\sigma k\right)+\frac{1}{2}m\alpha(\alpha+1)k^{-(\alpha+2)}k^2=0. \tag{84}$$

The terms relating to k^{-1} in (84) cancel each other out, and (84) simplifies to

$$k^{-\alpha}m\left(\rho-\alpha\sigma+\frac{1}{2}\alpha(\alpha+1)+\alpha m^{-\frac{1}{\alpha}}\right)=0. \tag{85}$$

The latter is satisfied if and only if $m=0$ or

$$m=\left(\frac{\alpha\sigma-\rho-\frac{1}{2}\alpha(\alpha+1)}{\alpha}\right)^{-\alpha}. \tag{86}$$

The optimal consumption rule is then given by

$$c(k)=\left(\frac{\alpha\sigma-\rho-\frac{1}{2}\alpha(\alpha+1)}{\alpha}\right)k, \tag{87}$$

i.e. a linear function of capital.[18]

3.3 Portfolio Selection

Our second example is based on a model that was first discussed by Eaton (1981) and presented in a slightly simplified form in Turnovsky (1995), where it is solved via use of HJB equations. We will use this example purely for illustration. Here, instantaneous macro-economic output dY and government spending dG are given as

$$dY = \alpha K(dt+dy) \tag{88}$$
$$dG = g\alpha K dt+\alpha K dz. \tag{89}$$

The constant α measures productivity, K denotes the capital stock, the only factor of production, and dy represents a productivity shock. Instantaneous government spending is expressed in terms of a fraction of output $g \in [0,1]$. The rate of return on government bonds dR_B is assumed to be

[18] For this to be well defined we require certain conditions on the parameters, that guarantee that the consumption capital ratio is positive, e.g. σ needs to be sufficiently large. We omit the details.

stochastic

$$dR_B = r_B dt + du_B, \tag{90}$$

where r_B and du_B are determined endogenously in macro-economic equilibrium. There is no money in this model, so real and nominal rates are identical. The marginal product of capital is determined as

$$dR_K = \frac{dY}{K} = \alpha dt + \alpha dy \equiv r_K dt + du_K. \tag{91}$$

Note that with a linear production technology, the marginal and average product of capital coincide.

The government's tax take is derived from taxation of income from physical capital:

$$dT = \tau r_K K dt + \tau' K du_K = \tau \alpha K dt + \tau' \alpha K dy. \tag{92}$$

Note that tax has a deterministic component and a stochastic component; according to Eaton (1981) it is possible to tax different components of income differently.

The stochastic optimization problem of the representative consumer is expressed as the choice of a consumption wealth ratio C/W and portfolio shares n_B and n_K in order to maximize

$$\mathbb{E}\left(\int_0^\infty \frac{1}{\gamma} C(t)^\gamma e^{-\beta t} dt\right), \tag{93}$$

subject to the wealth accumulation

$$\frac{dW}{W} = \left(n_B r_B + n_K(1-\tau) r_K - \frac{C}{W}\right) dt + dw, \tag{94}$$

with

$$dw \equiv n_B du_B + n_K(1-\tau') du_K. \tag{95}$$

Note that

$$n_B + n_K = 1 \tag{96}$$

and to simplify notation we use $n = n_B$ and $(1-n) = n_K$ in the following. Further note that dw can be written as

$$dw = \sigma(n) dZ, \tag{97}$$

where

$$\sigma(n)^2 = n^2\sigma_B^2 + (1-n)^2\sigma_K^2(1-\tau')^2 + 2n(1-n)(1-\tau')\sigma_{BK}, \tag{98}$$

and dZ is the increment of a Brownian motion Z. Using the notation of section 2, but with Z

replacing W (which is already used to denote the wealth process in this section), we have

$$\begin{aligned}
f(t,W,n,C) &= \frac{1}{\gamma}C^{\gamma}e^{-\beta t}, &(99)\\
b(t,W,n,C) &= W(nr_B+(1-n)(1-\tau)r_K - C, &(100)\\
\sigma(t,W,n,C) &= W\sigma(n). &(101)
\end{aligned}$$

We will derive the solution of this optimal control problem using equation (40) and (41) obtained from the Lagrangian approach. To do so one requires the partial derivatives of the functions in (99) - (101) with regards to the state variable W and the controls n and C. This is straightforward, the results are as follows:

$$f \quad \Rightarrow \quad \begin{cases} f_W = 0 \\ f_n = 0 \\ f_C = C^{\gamma-1}e^{-\beta t}, \end{cases} \tag{102}$$

$$b \quad \Rightarrow \quad \begin{cases} b_W = nr_B + (1-n)(1-\tau)r_K \\ b_n = W(r_B - (1-\tau)r_K) \\ b_C = -1, \end{cases} \tag{103}$$

$$\sigma \quad \Rightarrow \quad \begin{cases} \sigma_W = \sigma(n) \\ \sigma_n = W\sigma_n(n) \\ \sigma_C = 0. \end{cases} \tag{104}$$

Substituting these into (40) and (41) and suppressing all time indices for easier notation we obtain[19]

$$\begin{aligned}
-\frac{\mathbb{E}_t(d\lambda)}{dt} &= \sigma(n)\frac{d\lambda dZ}{dt} + \lambda\rho(n) &(105)\\
0 &= W\sigma_n(n)\frac{d\lambda dZ}{dt} + \lambda W\eta &(106)\\
0 &= C^{\gamma-1}e^{-\beta t} - \lambda, &(107)
\end{aligned}$$

with $\rho(n) = nr_B + (1-n)(1-\tau)r_K$ and $\eta = r_B - r_K(1-\tau)$. Solving equation (106) for $\frac{d\lambda dZ}{dt}$ and substituting into (84) as well as rearranging (86) gives

$$\begin{aligned}
-\frac{\mathbb{E}_t(d\lambda)}{dt} &= \lambda\left(\rho(n) - \frac{\sigma(n)}{\sigma_n(n)}\eta\right) &(108)\\
\lambda &= C^{\gamma-1}e^{-\beta t}. &(109)
\end{aligned}$$

[19]Note that (41) in fact results in two equations, one for each of the two control variables. This had been simplified in section 2 for the matter of illustration.

Denoting with $\mu = \frac{C}{W}$ the consumption to wealth ratio one obtains from (109), (94) and (97) that

$$\begin{aligned} d\lambda &= -\beta\lambda dt + (\gamma-1)\lambda[(\rho(n)-\mu)dt + \sigma(n)dZ] \\ &\quad + \frac{1}{2}(\gamma-1)(\gamma-2)\lambda\sigma(n)^2 dt, \end{aligned} \tag{110}$$

and from this

$$\frac{\mathbb{E}_t(d\lambda)}{dt} = (\gamma-1)\lambda\left[(\rho(n)-\mu) + \sigma(n)^2\left(\frac{\gamma}{2}-1\right)\right] - \lambda\beta. \tag{111}$$

Combining (111) with (108) gives

$$(\gamma-1)\lambda\left[(\rho(n)-\mu) + \sigma(n)^2\left(\frac{\gamma}{2}-1\right)\right] - \lambda\beta = -\lambda\left(\rho(n) - \frac{\sigma(n)}{\sigma_n(n)}\eta\right). \tag{112}$$

One immediately notes that the Lagrangian process λ cancels from (112) and that in consequence the latter is a purely algebraic equation. Using further that

$$\frac{d}{dn}\sigma(n)^2 = 2\sigma(n)\sigma_n(n) \Rightarrow \sigma_n(n) = \frac{\frac{d}{dn}\sigma(n)^2}{2\sigma(n)} \tag{113}$$

we obtain

$$(\gamma-1)\left[(\rho(n)-\mu) + \sigma(n)^2\left(\frac{\gamma}{2}-1\right)\right] - \beta = -\left(\rho(n) - 2\frac{\sigma(n)^2}{\frac{d}{dn}\sigma(n)^2}\eta\right). \tag{114}$$

Then, from (110) we conclude that

$$\frac{d\lambda dZ}{dt} = (\gamma-1)\lambda\sigma(n) \tag{115}$$

and therefore from equation (106)

$$(\gamma-1)\sigma(n)\sigma_n(n) + \eta = 0 \Rightarrow \frac{1}{2}(\gamma-1)\frac{d}{dn}\sigma(n)^2 + \eta = 0. \tag{116}$$

Substituting the latter into (114) we obtain

$$(\gamma-1)\left[\rho(n) - \mu + \sigma(n)^2\left(\frac{\gamma}{2}-1\right)\right] - \beta = -(\rho(n) + (\gamma-1)\sigma(n)^2). \tag{117}$$

Noticing that a number of terms cancel on both sides and solving for $\mu = \frac{C}{W}$ gives

$$\frac{C}{W} = \frac{\beta - \gamma\hat{\rho} - \frac{1}{2}\gamma(\gamma-1)\hat{\sigma}^2}{1-\gamma}, \tag{118}$$

where with the same notation as in Turnovsky (1995) $\hat{\rho}$ and $\hat{\sigma}^2$ denote the values of $\rho(n)$ and $\sigma(n)^2$ under the optimal portfolio policy n^*. Equation (118) is identical to equation 14.A.10a.) in Turnovsky (1995). However, with the Lagrangian approach its derivation becomes a purely algebraic exercise, avoiding any partial differential equations. This example can be further developed, however the purpose of this section is purely to illustrate our method, so we conclude at this stage.

4 Application to Numerical Schemes

In order to demonstrate the numerical potential of our approach we revert back to the discretized version of the scheme, specifically equation (33). By conditioning each of the factors that are multiplied with ξ_i or β_i on $\mathcal{F}_i$ and using the tower property of the conditional expectation we obtain

$$
\begin{aligned}
&\mathbb{E}\left(\sum_{i=0}^{n-1} f_x(i\Delta, X_i^*, \alpha_i^*)\Delta \cdot \xi_i + g_x(X_n^*) \cdot \xi_n + \sum_{i=0}^{n-1} f_a(i\Delta, X_i^*, \alpha_i^*)\Delta \cdot \beta_i\right) \qquad (119)\\
+&\mathbb{E}\left(\sum_{i=0}^{n-1}(\mathbb{E}(\lambda_{i+1}|\mathcal{F}_i) - \lambda_i)\xi_i + \left(\mathbb{E}(\lambda_{i+1}|\mathcal{F}_i)b_x\Delta + \sigma_x\mathbb{E}(\lambda_{i+1}\Delta W_i|\mathcal{F}i)\right)\xi_i - \lambda_n\xi_n\right.\\
&\left.\sum_{i=0}^{n-1}\left(\mathbb{E}(\lambda_{i+1}|\mathcal{F}_i)b_a\Delta + \sigma_a\mathbb{E}(\lambda_{i+1}\Delta W_i|\mathcal{F}_i)\right)\beta_i\right) = 0.
\end{aligned}
$$

As each of the coefficients in front of the ξ_i and β_i are $\mathcal{F}_i$ measurable, the fact that the ξ_i and β_i can be chosen from a complete system of $\mathcal{L}(\Omega, \mathcal{F}_i)$ and independent of each other implies that coefficients multiplied with each ξ_i resp. β_i add up to zero. This means that

$$\lambda_i = f_x\Delta + \mathbb{E}(\lambda_{i+1}|\mathcal{F}_i)(1 + b_x\Delta) + \mathbb{E}(\lambda_{i+1}\Delta W_i|\mathcal{F}_i)\sigma_x \qquad (120)$$

$$0 = f_a\Delta + \mathbb{E}(\lambda_{i+1}|\mathcal{F}_i)b_a\Delta + \mathbb{E}(\lambda_{i+1}\Delta W_i|\mathcal{F}_i)\sigma_a. \qquad (121)$$

Note that equations (120) and (121) are extension of equations (25) and (26) in Chow (1993) to the case where the volatility term is state and control dependent.[20] They are discrete time versions of equations (40) and (41) derived in section 2 and key to the derivation of numerical schemes. Chow (1993) tries to provide a justification as to why individual terms in the first order condition of his Lagrange function (24) should be zero, but is not mathematical rigorous. Our treatment of the Lagrange function and its derivatives in the Frechét sense, together with the argument presented above fills this gap. In addition, our treatment allows us to consider the important case, where the volatility term is state and control dependent.

In the following we will present a numerical scheme, which will compute Lagrange multipliers and optimal controls recursively. Unlike the scheme developed in Chow (1993, 1997) our scheme does not require linearization. While linearization along a deterministic steady state can lead to

[20] Chow (1993) discusses the case $x_{t+1} = f(x_t, u_t) + \epsilon_{t+1}$. Under this assumption, the volatility term σ is constant, hence σ_x and σ_a are zero. In equation (120) we have $(1 + b_x\Delta)$ and not $b_x\Delta$ as Chow (1993) would seem to suggest, but this is an effect of specifying the dynamics in terms of increments rather than absolute value, e.g. the analogous specification in Chow would be of the type $x_{t+1} = x_t + f(x_t, u_t) + \epsilon_{t+1}$.

good results for the optimal control along an optimal trajectory with a fixed initial condition, the computed controls are not reliable when a large shock effects the system, say as in the context of the sudden onset of a Financial crisis. The scheme presented here computes the optimal control as a global feedback control in line with the Lagrange multiplier method presented above.

Our scheme is based on equations (120) and (121) together with the discretized state dynamics

$$X_{i+1} \quad = \quad X_i + b(i\Delta, X_i, \alpha_i)\Delta + \sigma(i\Delta, X_i, \alpha_i)\Delta W_i, \tag{122}$$

but backwards in time. This means that starting with $\lambda_n = g_x(X_n^*)$ and given X_n^* on a discrete grid the dynamics (122) is simulated backward, that is (122) is solved for X_i given X_{i+1} and the shock ΔW_i possibly using a proxy for the control.[21] Then (120) and (121) are solved for λ_i where the conditional expectations in both equations are obtained by linearly regressing λ_{i+1} and $\lambda_{i+1}\Delta W_i$ on powers of the state variable X_i using the same generated random shocks as for solving (122). Typically the regression includes powers up to order 2 or 3. Instead of simple powers one can also use other series of polynomials, for example Hermite polynomials or series of elementary functions. One can also center the polynomials around a specific value, e.g. a deterministic steady state, to improve local properties of the solution.

Our approach shares similarities with the simulation of solutions of forward backward stochastic differential equations. As in our approach, the main difficulty in the numerical treatment of forward backward stochastic differential equations as compared to plain forward stochastic differential equations is to carry out the backward step in time, where at some point a conditional expectation needs to be taken. For forward backward stochastic differential equations, where the forward dynamics do not depend on the control this is slightly easier and can be done via Monte Carlo simulation and regression. Bouchard and Touzi (2004), Fahim, Touzi and Warin (2011) and Gobet, Lemor and Warin (2005) are important contributions in that context. In the case of a stochastic optimal control problem as discussed in this paper, an additional optimization needs to be carried out in this process and the conditional expectation to be taken in that backward step depends in general on the control process. This is reflected in our equation (121). Ludwig et al. (2012) present a numerical scheme to solve optimal stochastic control problems which also takes the continuous time equations (5), (7) and (9) as a starting point. In the backward step, they simply replace the conditional expectation of λ_{i+1} with the actual λ_{i+1}, which has been obtained in the previous backward step, compare equations (20) and (23) in Ludwig et al. (2012). This idea is adapted and implemented in our approach. Other relevant literature on this topic includes Tan (2014), who present an approach which is centered around the uncontrolled process X^0 which serves as an approximation to the optimally controlled process X, compare equations (2.6) and (2.7) in Tan (2014). Reiter (1999) and Longstaff and Schwartz (2001) were among the first to use regression based techniques to solve stochastic optimal control problems and Longstaff and Schwartz' (2001)

[21] This depends on the applications and sometimes need to be chosen to be an anticipative control, similar as in Ludwig et al (2012). If anticipative proxies are used, then they need to be regressed on powers of the state variable as well in an additional step. Our numerical example will illustrate this.

least square Monte Carlo approach has indeed become the benchmark for solving optimal stopping problems, a particular type of optimal control problem. However, the use of Monte Carlo based regression techniques in combination with the Lagrangian approach is entirely new and unique to our work.

Instead of providing general algorithmic details of our proposed scheme, we work out a specific example. This example expands on Chow's (1993) test-case (Section 5) and is also related to our example in section 3, but with slightly different parametrization.

$$\max \mathbb{E}\left(\int_0^\infty e^{-\rho t} c_t^\gamma dt\right), \tag{123}$$

$$dk_t = (\theta k_t^\alpha - c_t)dt + \sigma k_t dW_t. \tag{124}$$

Note that an explicit solution can be obtained for the case $\alpha = 1$, which is in fact the case that Chow (1993) uses for benchmarking his method. In this case the optimal control is explicitly given via

$$c_t = c(k_t) = q \cdot k_t, \text{with } q = \frac{\rho - \gamma\theta}{1-\gamma} + \frac{1}{2}\sigma^2\gamma. \tag{125}$$

Reiter (1997) also investigates the case for general α which we will also follow up. In this case, equations (120) and (121) become

$$\lambda_i = \mathbb{E}(\lambda_{i+1}|\mathcal{F}_i)(1 + \theta\alpha k_i^{\alpha-1}\Delta) + \mathbb{E}(\lambda_{i+1}\Delta W_i|\mathcal{F}_i)\sigma \tag{126}$$

$$c_i = \left(\frac{1}{\gamma} e^{\rho i \Delta}\mathbb{E}(\lambda_{i+1}|\mathcal{F}_i)\right)^{\frac{1}{\gamma-1}}, \tag{127}$$

and in each backward step k_i is obtained as the solution of

$$k_{i+1} = k_i + (\theta k_i^\alpha - \hat{c}_i)\,\Delta + \sigma\Delta W_i \tag{128}$$

with

$$\hat{c}_i = \left(\frac{1}{\gamma} e^{\rho i \Delta}\lambda_{i+1}\right)^{\frac{1}{\gamma-1}}, \tag{129}$$

i.e. the anticipative proxy of c_i, similar as in Ludwig et al (2012). In the case of $\alpha = 1$, i.e. Chow (1993), equation (128) can be solved explicitly for k_i, without much difficulty. For general α equation (128) needs to be solved numerically. As indicated before, the scheme operates backward according to the following schema[22]

$$
\begin{tikzcd}
\lambda_i & \begin{array}{c}\mathbb{E}(\lambda_{i+1}\Delta W_i|\mathcal{F}_i)\\ \mathbb{E}(\lambda_{i+1}|\mathcal{F}_i)\end{array} \arrow[l, "(126)"'] & \Delta W_i \arrow[l] \\
c_i & \hat{c}_i \arrow[l] \arrow[d, Rightarrow] \arrow[dr, <->] & \lambda_{i+1} \arrow[l, "(108)"'] \arrow[ul, "lr"'] \\
& k_i \arrow[ul, "lr"] & k_{i+1} \arrow[l, "(128)"']
\end{tikzcd}
$$

where the state variable at the final stage is modeled on a discrete grid and the final λ_n is obtained from the transversality condition (34) as proxy of an infinite time horizon transversality condition. This can be an iterative process. The two conditional expectations in the first row of the diagram are obtained as follows:

- λ_{i+1} and k_{i+1} are given as vectors from the previous round, each entry in the vector represents a different state (gridpoint) (typically 10000)
- corresponding to each state, one random sample $\Delta W_i \sim \mathcal{N}(0, \Delta)$ is generated (typically 10000) samples
- k_i is obtained as a vector from solving (128) (e.g. 10000 paths are generated backward in time)
- $\mathbb{E}(\lambda_{i+1}|\mathcal{F}_i)$ and $\mathbb{E}(\lambda_{i+1}\Delta W_i|\mathcal{F}_i)$ are obtained by regressing λ_{i+1} and $\lambda_{i+1}\Delta W_i$ on basis function of k_i, typically $1, k_i, k_i^2, k_i^3$ or a selection thereof or more suitable polynomial functions and series of elementary functions

Once this is done λ_i is obtained from (126) and c_i from regressing $\hat{c}_i$ on the same basis functions.

Testing our method for the case $\alpha = 1$ against the explicit solution (125) we obtain the relative errors presented in Table 1. Already after 1000 iterations the error is completely negligible, after 5000 iterations it is practically zero. The number of paths generated is set to 10000. Under these settings the elapsed time for 5.000 iterations is 28.392631 sec on an 11th Gen Intel(R) Core(TM) i5-11400 @ 2.60GHz, 2592 Mhz, with 6 kernels and 12 logical processors.

The precision of our method is hence similar to Chow (1993), compare page 750 and Table 2. However, as Reiter (1997) demonstrated Chow (1993) runs into trouble if the model becomes truly non-linear, e.g. for the case $\alpha \neq 1$. For our approach non-linearity does not impose great difficulties, at least for our benchmark example (123) and (124). The following figure shows the optimal consumption to capital ratio obtained from regressing on k_i:

For $\alpha = 1$ the optimal consumption capital ratio coincides with the results represented by table 1 and is extremely close to the exact value of $q = 0.7$ obtained from (125). For $\alpha \neq 1$ it is not possible to solve the backward equation (128) explicitly. Instead this equation is solved numerically,

[22] Numbers in brackets refer to the corresponding equations in the main text, lr indicates linear regression.

iterations	c/k rel error
50	-0.154
100	-0.91
500	0.0051
1000	-2.5768e-04
5000	-9.6659e-05

Table 1: Relative error in the computation of the optimal consumption to capital ratio relative to the exact solution (125) for the case $\alpha = 1$.

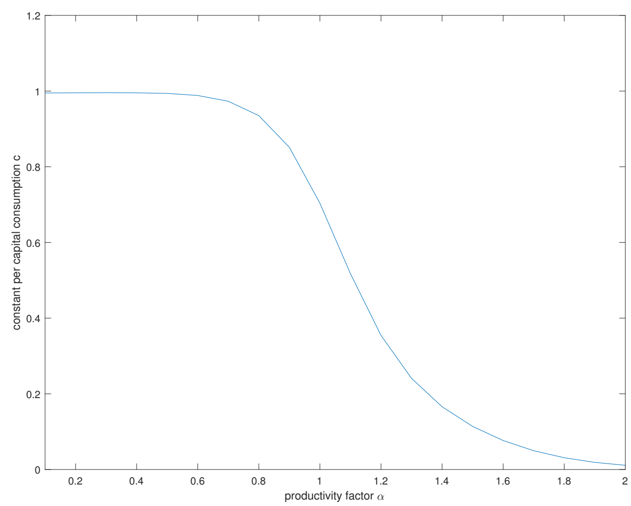

Figure 1: Optimal consumption to capital ratio obtained from regressing on k_i as a function of productivity coefficient α. Parameters: $\Delta = 0.01; \sigma = 0.01; \rho = 0.5; \gamma = 0.5; \theta = 0.3; n = 500, m = 1000$.

using the matlab fzero command. This is highly accurate but slows down the algorithm by roughly a factor of 10. To compensate for that we reduced the number of paths generated to 1000 and the number of iterations to 500. In addition the optimal control needs to be calculated for each $\alpha = 0.1, 0.2, ...1.9, 2.0$. The computational time for generating Figure 1 is roughly 5 minutes, which is still extremely fast. The figure shows that the consumption to capital ratio falls with increasing productivity, which is in line with standard economic intuition, where consumption decreases with marginal productivity (which in equilibrium reflects the interest rate). We abstain from further economic interpretation of the results, which are not the subject of this paper.

While our numerical scheme has been derived and motivated from within a continuous time setting, the Lagrange approach initially presented in section 2 covers discrete time stochastic optimal control problems with continuous state variables. Equations (120) and (121) are in fact in discrete time. As such our approach also offers potential for Monte Carlo and regression based numerical schemes to be derived for models that were initially formulated in discrete time, which in fact was the initial focus of Chow (1992) (equations (1), (2) and (3)). It is fair to say that the use of linear regression and Monte Carlo backward simulation can be computationally intensive for higher dimensional examples. We expect that our method can be improved and extended by using recent methodology from machine learning, specifically deep learning and artificial neural networks. We leave this for future work.

5 Conclusions

We have demonstrated how the classical Lagrange approach known from finite dimensional calculus and introductory Economics lectures can be extended and used to solve continuous time stochastic optimal control problems. This agenda was pioneered by Chow and co-authors with the publications in Chow (1992, 1993) and Kwan and Chow (1997). Our works goes beyond their results showing how the relevant equations directly transcend from the Lagrangian multiplier approach in a mathematical rigorous way using results from infinite dimensional functional analysis. We also show how this approach is embedded in more modern approaches such as the stochastic maximum principle expressed within the language of backward stochastic differential equations. Our two key equations, (40) and (41), however allow us to identify the Lagrangian λ_t without any reference to backward stochastic differential equations, making this approach far more accessible. In addition we demonstrate how these two equations can be employed to derive efficient numerical schemes to solve stochastic optimal control problems using backward Monte Carlo simulation and regression based techniques. We provide a number of examples from Economics, which demonstrate the simplicity and efficiency of our approach in practice. The examples are both of theoretical and numerical in nature and show that the method can be applied effectively, leading to highly accurate results. In conclusion, our paper shows that 30 years on since Chow (1992) introduced the Lagrangian method for solving optimal control problems, this methodology is still highly relevant and timely having its own place within more recent developments within the mathematical literature on stochastic optimal control theory.

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